ON SEMI/TAME CLASSES OF FINITE STRUCTURES

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Abstract. We modify the definition of robust chains from [25] as an approach to formalizing the Finite Sub-model Property (see [21], for example) from the perspective of a class of finite structures with a generic model (i.e. having the Amalgamation and Joint-embedding properties). We characterize this formulation, which we call super-robustness, *modulo* a weaker condition (that is closer to the the situation of a robust chain) that we call robustness. After observing that the more natural condition of a given class is super-robustness *up to cofinality* – which we call tameness – we prove that a number of natural constructions preserve tameness, and we prove some necessary conditions for tameness refining strong-minimality and total-categoricity. We define a notion of "locally pro-generic" type and verify that this definition provides a re-formulation of þ-independence for generic models of certain tame classes. Finally, we show that any class that has the 0,1-law for first-order sentences compatibly with its generic theory is a tame class.

1. Introduction

From the standpoint of infinitary model theory, when given a class **C** of finite structures, the natural reflex is to compile **C** into a single infinite structure and hope for the best – that is, hope that the analysis of the infinite compiled structure has some bearing on the members of **C**.

There are (at least) two important ways to make this compilation: either one works with a generic model of the class obtained through a construction invented by R. Fraïssé [10], or one works with an ultraproduct over the class. The former approach yields an infinite structure with a relatively concrete, "smooth" model theory; indeed, the Rado graph, infinite-dimensional vector spaces over finite fields (with or without additional Lie structure), and algebraically-closed fields of positive characteristic can all be encountered in this manner, and all of these have a fairly well understood model theory. However, it can be quite difficult to apply the results of analyses of these generic structures to members of the classes from which they derive. On the other hand, to understand ultraproducts of finite structures (i.e. pseudo-finite theories) generally demands much more subtle analyses, but the results push down to finite structures relatively easily. The techniques of [11, 19, 33], for example, may suggest the degree of subtlety required (but more immediately, one would probably expect that working with algebraically-closed fields is easier than working with infinite quasi-finite pseudo-algebraically-closed fields). An ideal situation, then, is one in which analyses can be carried out on some sort of Fraïssé limit or generic model that providentially turns out to be elementarily equivalent to an ultraproduct over the same class, allowing the results to push down more or less immediately. Previously, this ideal situation has been observed from the standpoint of an infinite (usually $\aleph_0$-categorical) structure under the name of the *Finite Sub-model Property*, as in [21] for example, but we will approach the same idea in a slightly different way – from the standpoint of the class of finite "approximating" structures.

Implicit in this ideal situation is the requirement that not only first-order sentences transfer back and forth between between **C**, its ultraproducts, and its generic model but *all definable sets.* Viewed from the perspective of **C** alone, the ideal situation requires that formulas – whose interpretations ordinarily can vary wildly even along a chain of finite structures – must stabilize in some sense relative to varying members of **C**. To formalize this perspective, we modify and build upon the definition of *robust chain* examined in [25], arriving at our definitions of *robust and super-robust classes*.

When $\mathbf{C}$ is super-robust, we find ourselves with a workable interpretation of the analogy

$$\text{generic/limit of } \mathbf{C} : \text{members of } \mathbf{C} :: \text{monster model of } T : \text{small models of } T$$

which is an essential precondition for the kind of analysis carried out in [14] and [12]. Building on the result of the former, in this article, we uncover a distinctly topological re-formulation of þ-independence and rosiness in certain super-robust classes by considering a slightly novel notion of locally pro-generic type (which is closely related to a notion used in [13]). Examining the phenomenon of super-robustness relative to its weaker cousin, robustness, we uncover a number of alternative characteristic properties of super-robustness, including a well-quasi-ordering condition and a way of viewing the Finite Sub-model Property (for the generic model) through appropriate ultrafilters (suggesting, we think, that the Finite Sub-model Property may also have topological formulation in an appropriate space of ultrafilters). We also examine a number of constructions of new classes of finite structures from old – such as direct products and direct sums which play important roles in stating and proving so-called algorithmic meta-theorems (see [22]) – and we find that having made the idea of "adherence" of definable sets a precise notion in the form of super-robustness, it is not difficult to show that that such constructions preserve this property. Finally, after formulating an apparently weaker form of 0,1-law than the standard one, we show that for any class $\mathbf{C}$ that has a 0,1-law for first-order sentences whose asymptotically almost-sure theory coincides with its generic theory, $\mathbf{C}$ is super-robust *up to cofinality.*

In a departure from the work of [21], many of our results, and the techniques of their demonstrations, pertain to structures that need not be $\aleph_0$-categorical. Thus, our preservation results apply to broad classes of uncountably-categorical theories (including some arising from Hrushovski constructions and similar), and our result on 0,1-laws is informative regarding "sparse" 0,1-laws as proved in [31] and examined further in [3, 4, 29] and elsewhere.

In most cases of interest, the situation of super-robustness *up to* extracting a cofinal subclass seems to be the best possible – even for the class of all finite graphs (whose generic model is the Rado graph), there are "bad" graphs of all cardinalities. Thus, in many of our results our goals are to identify necessary or characteristic conditions for super-robustness up to cofinality, which we call *tameness or semi-tameness* depending on whether that class' generic theory is $\aleph_0$-categorical.

Outline of the article. The article is organized as follows:

- **Section 2:** We recall, beginning with a discussion of Fraïssé classes and Fraïssé's construction, the general setting of classes of finite structures that *have* a generic model, and we identify three families of classes – Fraïssé, coherent, and semi-coherent classes in ascending order of generality – that will occupy us for the rest of the article. We also discuss axiomatic strong-substructure relations (which are the end-all and be-all of the Hrushovski and Spencer-Shelah constructions), showing that classes equipped with these can be fruitfully analyzed in the previous context, in which only the ordinary induced substructure relation is permitted, provided that we allow changes of language. Finally, we review the topological (Baire categorical) re-formulation of Fraïssé's construction, though we use this setting only as a loose template for our considerations of þ-independence in Section 5.
- **Section 3:** We formalize the idea of "adherence of definable sets" in the forms of robust and super-robust classes. We then proceed to characterize super-robustness *modulo* robustness in several rather diffferent-looking ways. As a result, we recover a necessary condition for semi/tameness that we will use in the proof of the result on 0,1-laws in Section 6.
- **Section 4:** We demonstrate (following [30] very closely) that certain kinds of strongly-minimal theories arise as generic models of semi-tame classes. Extending this result using well-known

facts about uncountably-categorical theories (but eschewing the difficult geometric analysis of [34]), we sketch a proof of the fact that totally-categorical theories invariably arise as generic models of semi-tame classes. We also use a version of the Feferman-Vaught theorem on the theories of sums and products of structures to show that semi/tameness is preserved under taking direct products and direct sums of classes (which we also define).

Section 5: We examine some "stability-theoretic" notions in the context of tame classes. In particular, we show that the generic theory of a tame class is necessarily NSOP (so as a corollary, NIP=stable for such theories). After that, we define a notion of "locally pro-generic type," building on some insights of [20], and we use this notion (along with a result from [14]), to provide a topological formulation of þ-independence for the generic theory T of a tame class (when it eliminates imaginaries).

Section 6: We define a version of the 0,1-law for first-order sentences that is compatible with [semi-]coherent classes that are not Fraïssé classes and that is particularly compatible with the ultrafilter characterization of (a strong form of) the Finite Sub-model Property identified in Section 3. We then prove that classes that exhibit this 0,1-law* compatibly with their generic theories are necessarily [semi-]tame.

Before delving into the material of this rather-long paper, we provide a few examples and non-examples of [semi-]tame classes for the reader to hold in mind as we go. The reader will see that the tameness of several of these examples is either easily verified or known from previous work. Some of these facts are novel, however, and the reader will have to wait until later sections for verification.

Examples:

- The class of all of finite graphs, whose generic model is the Rado graph.
- After fixing a finite field F, that class of all finite vector spaces over F.
- Given a Lie-coordinatizable structure $\mathcal{M}$, the class of all structures isomorphic to envelopes of $\mathcal{M}$ – see [7]. (Thus, the previous example is generalized to allow sufficiently generic Lie structure.)
- Strong substructures of the generic model involved in the proof of the 0,1-law for sparse graphs after fixing an irrational number $0 < \alpha < 1$ for specification of sparsity. (See Section 6.)
- Suitably formulated "envelopes" of strongly-minimal structures that may *not* be $\aleph_0$-categorical but *do* have enough uniformly-local-finiteness (for the algebraic closure operator) to carry out a certain argument given in Section 4.

Some non-examples of [semi-]tame classes are the following:

- The class of finite cycle graphs, the class of finite path graphs, or any class of trees of unbounded height.
- For $p > 0$, the class of finite fields of characteristic p. (One way of demonstrating that such a class of fields is not semi-tame is to use the ultrafilter characterization of super-robustness. We must also note that non-semi-tameness obviously does not preclude arguments that are, in some sense, inspired but properties of semi-tame classes.)

Not surprisingly, there are several interesting classes of finite structures whose status with respect to [semi-]tameness is not yet known:

- The class of triangle-free finite graphs, and more generally, the classes of K_n-free finite graphs for arbitrary n that generate the Henson graphs.
- Infinite "direct sums": Classes obtained from two given classes $\mathbf{C}_0, \mathbf{C}$ by taking members $\mathfrak{a}$ of $\mathbf{C}_0$ and "gluing" members of $\mathbf{C}$ to each member of the universe of $\mathfrak{a}$.

2. Generalizations of Fraïssé classes

In this section, we first establish some basic notation and conventions, and then we define several families of classes of finite structures, two of which will be the primary focus of the rest of the paper. These families are obtained by relaxing or completely discarding the *Heredity Property* (HP) from the definition of the older notion of Fraïssé classes, so we take up a whole subsection recalling this definition and the all-important Fraïssé construction of the generic model. Thereafter, we introduce the notions of semi-coherent and coherent classes, in which HP is, respectively, discarded entirely or relaxed to a *Weak Löwenheim-Skolem Property* (WLSP) that characterizes $\aleph_0$-categoricity of the generic model.

In order to accommodate certain very well-known constructions – for exotic strongly-minimal structures and "sparse" 0,1-laws – it is necessary to extend our framework some way beyond what can be accomplished with the ordinary (induced) substructure relation. This leads us to a discussion of "abstract" coherent and semi-coherent classes, in which axiomatic strong-substructure relations replaces the ordinary substructure relation. In this discussion, we will see that if one permits changes of language (as we do), then this additional generality does not lend us any great new power relative to the goals of this article.

Finally, we briefly review the topological setting for the previous discussion that justifies the term "generic model" (where many authors use the term "Fraïssé limit"). In Section 5, we will use a variation of this topological setting to develop an alternative formulation of þ-independence in a coherent class.

2.0.1. *Notation for classes of finite structures.* For use throughout this paper, we establish a few conventions about the languages in play, the notation for finite and possibly-infinite structures, and the requirements for justly asserting "$\mathbf{C}$ is a class of finite structures."

Convention (First language of convention). Throughout this article, any language $\mathscr{L}$ in question is the first-order language built over a countable signature $\mathrm{sig}(\mathscr{L})$ that has *no function symbols and only finitely many constant symbols.*

Convention. Fix a language $\mathscr{L}$.

- $\mathscr{L}$-structures that *may be* infinite are denoted by uppercase calligraphic letters, $\mathcal{M}, \mathcal{N}, ...$ and so forth, and their universes are denoted by the corresponding uppercase Roman character, $M, N, ...$, respectively, with cardinalities $|M|, |N|, ...$ respectively.
- If $\mathcal{M}$ is an $\mathscr{L}$-structure and $A \subseteq M$, then $\mathcal{M}[A]$ denotes the induced substructure of $\mathcal{M}$ with universe, $A \cup \{c^{\mathcal{M}} : c$, a constant of $\mathrm{sig}(\mathscr{L})\}$. Accordingly, we write $\mathcal{M} \leq \mathcal{N}$ to mean that $M \subseteq N$ and $\mathcal{N}[M] = \mathcal{M}$ – i.e. the inclusion map $inc_M : M \to N$ is an embedding of $\mathscr{L}$-structures. Extending this notation somewhat, we write $\mathcal{M} \leq^* \mathcal{N}$ to mean that there is an embedding $\mathcal{M} \to \mathcal{N}$. (The symbols $\preceq$ and $\preceq^*$ may understood similarly.)
- We use lowercase fraktur letters, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, ...$ and so on, to denote $\mathscr{L}$-structures that *are certainly finite.* The universe of $\mathfrak{a}$ is denoted $\|\mathfrak{a}\|$, and $|\mathfrak{a}|$ is the cardinality of $\mathfrak{a}$ – i.e. $|\mathfrak{a}| = \big|\|\mathfrak{a}\|\big|$. (This just saves us from writing, " where ... is a finite structure" *ad nauseum.*) The notations already mentioned pertain as well to finite structures.

Convention. To say, "$\mathbf{C}$ is a class of finite structures," we require:

- All members of $\mathbf{C}$ are finite structures for the same language $\mathscr{L}$.
- $\mathbf{C}$ is closed under isomorphism: If $\mathfrak{a} \in \mathbf{C}$ and $\mathfrak{b} \cong \mathfrak{a}$, then $\mathfrak{b} \in \mathbf{C}$.
- $\mathbf{C}$ is infinite *modulo* isomorphisms.
- For all $0 < n, N < \omega$, the set $\{\mathrm{qftp}^{\mathfrak{a}}(\bar{b}) : \mathfrak{a} \in \mathbf{C}, \bar{b} \in \|\mathfrak{a}\|^n, |\mathfrak{a}| \leq N\}$ is finite. Here, $\mathrm{qftp}^{\mathfrak{a}}(\bar{b})$ denotes the quantifier-free-complete type of $\bar{b}$ in the sense of $\mathfrak{a}$. (Coupled with the first language convention, we call this "the" language convention in a few places in this paper.)

Given some structure $\mathcal{N}$ (possibly finite), we write $\mathbf{C}(\mathcal{N})$ for the set $\{\mathfrak{a} \in \mathbf{C} : \mathfrak{a} \leq \mathcal{N}\}$.

Definition 2.1. When $\overline{x} = (x_0, ..., x_{n-1})$ is an n-tuple (of variables or elements of a structure, say) and $S \subseteq n$, we write $\overline{x}_{\restriction S}$ for the sub-tuple $(x_{i_0}, ..., x_{i_{m-1}})$, where $S = \{i_0 < \cdots < i_{m-1}\}$.

2.1. **Fraïssé classes.** The various kinds of classes of finite structures that we will discuss in this paper are all perturbations of the old notion of Fraïssé classes. Therefore, we will spend this subsection recalling the definition of a Fraïssé class, the construction of the generic model, or Fraïssé limit, of a Fraïssé class, and a few other related ideas. (In one of the most important cases in this article, that of coherent classes, it is true that "Morleyization" always yields a Fraïssé class, but this is not true of semi-coherent classes.) Essentially all of the material in this subsection can be found in much greater detail in [17].

Definition 2.2. Let $\mathbf{C}$ be a class of finite structures. We say that $\mathbf{C}$ is a Fraïssé class if it has the following three properties:

- **Joint-embedding (JEP):** For any two $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathbf{C}$, there are $\mathfrak{b} \in \mathbf{C}$ and embeddings $\mathfrak{a}_0 \to \mathfrak{b}$ and $\mathfrak{a}_1 \to \mathfrak{b}$.
- **Amalgamation (AP):** For $\mathfrak{a}, \mathfrak{b}_0, \mathfrak{b}_1 \in \mathbf{C}$ and embeddings $f_i : \mathfrak{a} \to \mathfrak{b}_i$ $(i < 2)$, there are $\mathfrak{c} \in \mathbf{C}$ and embeddings $f_i' : \mathfrak{b}_i \to \mathfrak{c}$ $(i < 2)$ such that $f_0' \circ f_0 = f_1' \circ f_1$.
- **Heredity (HP):** $\mathfrak{a}[B] \in \mathbf{C}$ whenever $\mathfrak{a} \in \mathbf{C}$ and $B \subseteq \|\mathfrak{a}\|$.

One can (and perhaps should) think of AP and JEP as precisely the properties of $\mathbf{C}$ that make the following construction of a countably infinite structure possible. In our statement of Fraïssé's construction, JEP is applied at every odd stage, and AP is applied at every positive even stage. We note that HP has nothing to with this construction, and its role comes to the fore only in Theorem 2.5 below. HP is also irrelevant to the characterization, Theorem 2.3, of the formal generic model that is the result of Fraïssé's construction.

Fraïssé's construction.

Setup: We are given a Fraïssé class $\mathbf{C}$ and a countably infinite set M. Let $\mathbf{C}[M]$ be the set of structures $\mathfrak{a} \in \mathbf{C}$ such that $\|\mathfrak{a}\| \subset M$, and let $\mathfrak{a}_0, \mathfrak{a}_1, ..., \mathfrak{a}_n, ...$ be an enumeration of $\mathbf{C}[M]$. Also, let $f_0, f_1, ..., f_n, ...$ be an enumeration of all embeddings $\mathfrak{a}_i \to \mathfrak{a}_j$ $(i, j < \omega)$, where each f_n is understood as just a finite subset of $M \times M$. Finally, let $\langle , \rangle : \omega \times \omega \to \omega$ be some bijection, and let $\langle x, y, z\rangle$ be shorthand for $\langle \langle x, y\rangle , z\rangle$ (which is a bijection $\omega{\times}\omega{\times}\omega \to \omega$).

The construction is in ω stages.

- **Stage 0:** Let $\mathfrak{b}_0 = \mathfrak{a}_0$.
- **Stage $s + 1 = 2\langle n, i\rangle + 1$:** Choose $m < \omega$ such that $\mathfrak{b}_s \leq \mathfrak{a}_m$ and $\mathfrak{a}_n \leq^* \mathfrak{a}_m$; then set $\mathfrak{b}_{s+1} = \mathfrak{a}_m$.
- **Stage $s + 1 = 2\langle m, n, i\rangle + 2$:** Suppose f_n is an embedding $\mathfrak{a}_j \to \mathfrak{a}_k$ where $\mathfrak{a}_k \leq \mathfrak{b}_s$, and suppose $\mathfrak{a}_j \leq \mathfrak{a}_m$. Then, choose $p, q < \omega$ such that $\mathfrak{b}_s \leq \mathfrak{a}_p$ and $f_q \supseteq f_n$ is an embedding $\mathfrak{a}_m \to \mathfrak{a}_p$, and then set $\mathfrak{b}_{s+1} = \mathfrak{a}_p$. (If either of these hypotheses fails, then just set $\mathfrak{b}_{s+1} = \mathfrak{b}_s$.)

In the end, we set $\mathcal{M} = \bigcup_{s<\omega} \mathfrak{b}_s$, which we call a *generic model of* $\mathbf{C}$.

Just what sort of thing Fraïssé's construction actually produces is characterized in the following proposition.

Proposition 2.3. *Let $\mathbf{C}$ be a class with both AP and JEP (possibly but not necessarily a Fraïssé class). Let $\mathcal{M}$ be the result of Fraïssé's construction.*

1. *($\mathbf{C}$-Universality) For every $\mathfrak{a} \in \mathbf{C}$, there is an embedding $\mathfrak{a} \to \mathcal{M}$.*
2. *($\mathbf{C}$-Homogeneity) For every $\mathfrak{a} \in \mathbf{C}$ such that $\mathfrak{a} \leq \mathcal{M}$, for every embedding $f : \mathfrak{a} \to \mathcal{M}$, there is an automorphism $g \in Aut(\mathcal{M})$ such that $f \subseteq g$.*
3. *For every $A \subset_{fin} M$, there is a finite substructure $\mathfrak{b} \leq \mathcal{M}$ such that $\mathfrak{b} \in \mathbf{C}$ and $A \subseteq \|\mathfrak{b}\|$.*

4. *Up to isomorphism, $\mathcal{M}$ is the unique countable structure that satisfies items 1-3. In particular, if $\mathcal{M}'$ is the result of Fraïssé's construction using possibly different enumerations and bijections $\omega \times \omega \to \omega$, then $\mathcal{M}' \cong \mathcal{M}$.*

   Thus, we may speak of the *generic model of a class* $\mathbf{C}$ *that has AP and JEP, and the assignment $T_{\mathbf{C}} := Th(\mathcal{M})$ is well-defined.*
5. *$\mathcal{M}$ is the prime model of $T_{\mathbf{C}}$.*

Fact 2.4. Let $\mathbf{C}$ be a class with both AP and JEP (possibly but not necessarily a Fraïssé class). Then $T_{\mathbf{C}}$ is model-complete. (To see this, note that if q is the quantifier-free-complete type of a tuple $\overline{a}$ that enumerates some $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$, then q isolates the complete type of $\overline{a}$ because $\mathcal{M}$ is $\aleph_0$-homogeneous. Since every finite subset of $\mathcal{M}$ extends to the universe of some $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$, existential formulas suffice. When a "strong-substructure" relation replaces the ordinary substructure relation, the situation becomes somewhat more complicated – see [3].)

As mentioned above, HP plays no role in Fraïssé's construction as such, but it does have a role to play. Namely, against the background of AP and JEP, it characterizes the conjunction of $\aleph_0$-categoricity and elimination of quantifiers. (See [17].)

Theorem 2.5. *Let T be a complete theory in a countable language (satisfying our language convention), and let $\mathcal{M}$ be a countable model of T. The following are equivalent:*

1. *T is $\aleph_0$-categorical and eliminates quantifiers.*
2. $\mathrm{Age}(\mathcal{M}) := \{\mathfrak{a} : \exists$ an embedding $\mathfrak{a} \to \mathcal{M}\}$ *is a Fraïssé class, and T is the theory of the its generic model.*

2.2. **Losing or relaxing the Heredity Property: semi/coherent classes.** We have noted that Fraïssé's construction works perfectly with any class $\mathbf{C}$ of finite structures that has both AP and JEP, producing a structure that is unique up to isomorphism. We also saw the power that the additional property HP has on the theory of the generic model of a Fraïssé class. In this subsection, we look a little closer at what is true of the generic model and its theory when HP is relaxed or discarded altogether.

Definition 2.6. Let $\mathbf{C}$ be a class of finite structures. We say that $\mathbf{C}$ has the *Weak Löwenheim-Skolem Property* (WLSP) if there is a function $\lambda : \omega \to \omega$ such that for all $\mathfrak{a} \in \mathbf{C}$ and $X \subseteq \|\mathfrak{a}\|$, there are $\mathfrak{b}, \mathfrak{c} \in \mathbf{C}$ such that $\mathfrak{a} \leq \mathfrak{c}$, $\mathfrak{b} \leq \mathfrak{c}$, $X \subseteq \|\mathfrak{b}\|$, and $|\mathfrak{b}| \leq \lambda(|X|)$.

Definition 2.7. Let $\mathbf{C}$ be a class of finite structures.

- $\mathbf{C}$ is a semi-coherent class if it has AP and JEP.
- $\mathbf{C}$ is a coherent class if it is semi-coherent and has WLSP.

Thus, Fraïssé's construction and the characterization given in Proposition 2.3 pertain precisely to semi-coherent classes, and Fraïssé classes are precisely those semi-coherent classes that have HP. The following theorem, which is due to [23], shows that in some sense, coherent classes lie between the two extremes.

Theorem 2.8. *Let $\mathbf{C}$ be a semi-coherent class. The following are equivalent:*

(1) $\mathbf{C}$ *is coherent.*
(2) $T_{\mathbf{C}}$ *is $\aleph_0$-categorical.*

To summarize the characteristics of our three kinds of interesting classes of finite structures, we include the following table (also basically due to [23]).

Type	*Class properties*	*Gen. theory's properties*
Semi-coherent	AP, JEP	$T_{\mathbf{C}}$ exists and is model-complete
Coherent	AP, JEP, WLSP	$T_{\mathbf{C}}$ exists, is model-complete, and is $\aleph_0$-categorical
Fraïssé	AP, JEP, HP	$T_{\mathbf{C}}$ exists, eliminates quantifiers, and is $\aleph_0$-categorical

There are a few reasons for introducing this new terminology and focusing attention on the dichotomy of coherent versus semi-coherent – rather than Fraïssé versus semi-coherent classes. For one thing, it's certainly a little easier both to say out loud and to typeset "semi-coherent class" than "class of finite structures with AP and JEP."

More importantly, coherence is invariant under a number of changes to the context that are either considered natural in the model theory of infinite structures or just *are* more natural for some of the analyses carried out later in this paper. (The fact of being a Fraïssé class is *not* necessarily invariant in these ways).

- Adding constants: Let $\mathbf{C}$ be a coherent class, and let $\mathcal{M}$ be its generic model. Then let $\overline{a} = (a_0, ..., a_{n-1})$ be a finite tuple of elements of M, and let $\mathscr{L}(\overline{c})$ be the expansion of the language $\mathscr{L}$ of $\mathbf{C}$ obtained by adding new constant symbols $c_0, ..., c_{n-1}$.

  Let $\mathbf{C}'$ be the class of finite $\mathscr{L}(\overline{c})$-structures $\mathfrak{b}$ such that $\mathfrak{b}\restriction\mathscr{L} \in \mathbf{C}$ and $\mathrm{qftp}^{\mathfrak{b}\restriction\mathscr{L}}(c_0^{\mathfrak{b}}, ..., c_{n-1}^{\mathfrak{b}}) = \mathrm{qftp}^{\mathcal{M}}(\overline{a})$. Then $\mathbf{C}'$ is a coherent class with generic model $(\mathcal{M}, \overline{a})$.

- Language invariance: Let $\mathbf{C}$ be a coherent class in a language $\mathscr{L}$ with generic model $\mathcal{M}$. Let $\mathscr{L}'$ be some other language, and let $\mathcal{M}'$ be some $\mathscr{L}'$-structure with universe $M' = M$.

  Suppose that every quantifier-free 0-definable set of $\mathcal{M}'$ is also quantifier-free 0-definable in $\mathcal{M}$ and vice versa. Then $\mathrm{Age}(\mathcal{M}')$ contains a coherent class whose generic model is $\mathcal{M}'$.

- Cofinality invariance: For two classes $\mathbf{C}_1, \mathbf{C}_2$ of finite structures in the same language $\mathscr{L}$, we will say that $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are cofinal in each other* if (i) for every $\mathfrak{a} \in \mathbf{C}_1$, there is a $\mathfrak{b} \in \mathbf{C}_2$ such that $\mathfrak{a} \leq \mathfrak{b}$, and (ii) for every $\mathfrak{a} \in \mathbf{C}_2$, there is a $\mathfrak{b} \in \mathbf{C}_1$ such that $\mathfrak{a} \leq \mathfrak{b}$.

  Noting that if $\mathbf{C}_1$, $\mathbf{C}_2$ are cofinal in each other, then $T_{\mathbf{C}_1} = T_{\mathbf{C}_2}$, one finds that if they are cofinal in each other and $\mathbf{C}_1$ is coherent, then $\mathbf{C}_2$ is also coherent. (Observed in [23].)

The dichotomy between coherence and semi-coherence is interesting because nature has decreed that a number of important model-theoretic constructions are associated with coherent classes and some with semi-coherent classes that are not coherent. Among the former, we find the class of all finite graphs (giving rise to the random graph, or Rado graph, as the generic model) as well as the classes of finite linear, affine, and projective spaces over a fixed finite field F, possibly with additional Lie-geometric structure. Of course, the latter are the key building blocks of the structure theory developed in [6] and, when some additional Lie structure is allowed, in [7]. We also observe that in the geometric examples, if we observe our language convention – in which function symbols are not permitted – then in their natural languages, these need not be Fraïssé classes.

On the other hand, several of the most interesting and finely-tailored constructions in model theory give rise to strictly semi-coherent classes or even something more general. Indeed, one might consider E. Hrushovski's construction of an exotic strongly minimal theory [18] and the analysis in [31] that identifies the asymptotic theory of random sparse graphs (i.e. having edge probability $n^{-\alpha}$ for irrational $\alpha \in (0, 1)$) with the generic theory of the same. In these cases, the generic model is not $\aleph_0$-categorical, so no alteration of the language in play could possibly yield a coherent class.

Furthermore, in both cases, the sparsity conditions (associated with pre-dimension and strong-substructure relations) very obviously thwart WLSP. Yet another, and possibly simpler, example of a strictly semi-coherent class is that of all finite fields of characteristic p, where $p > 0$ is fixed; here, the generic model is just the algebraic closure $\overline{F_p}$ of the prime field F_p, and WLSP is thwarted by the fact that one can find elements of $\overline{F_p}$ of arbitrarily high degree over F_p. Since one of the main contributions of this paper involves 0,1-laws, it seems obligatory to extend the context of the discussion far enough to include these important examples.

As hinted at above, the context of semi-coherent classes does not fit exactly with any of the constructions that use the technology of $\mathbb{R}$-valued pre-dimensions. This is due to the fact that the associated strong-substructure relation is not first-order. For the later analyses in this paper to go through, therefore, it may be necessary to make a change of language, replacing an "abstract semi-coherent class" with a genuine semi-coherent class. We discuss this kind of transformation in the next subsection. (While it is possible to make sense of 0,1-laws *modulo* such a change of language, it seems necessary to alter the manner in which asymptotic probabilities are computed, obtaining the notion of 0,1-laws* for first-order logic (see Section 6). At the moment, it appears that 0,1-laws* may be strictly more general than 0,1-laws.)

2.3. **Axiomatic strong-substructure relations.** As just mentioned, many of the classes of finite structures that have been useful in model-theoretic constructions previously – as in [3, 4, 18, 32, 8, 2], in no particular order – are organized not by the ordinary substructure relation but by some more subtle strong-substructure relation. For example, the first step in the project of [18] is to define a pre-dimension,

$$\delta(\mathfrak{a}) = |\mathfrak{a}| - |R^{\mathfrak{a}}|/6$$

on the class of finite structures $\mathfrak{a}$ in the language over the signature $\{R^{(3)}\}$ with just one 3-ary relation symbol (further requiring that $R^{\mathfrak{a}}$ is symmetric – $(a_0, a_1, a_2) \in R^{\mathfrak{a}} \Rightarrow (a_{\sigma(0)}, a_{\sigma(1)}, a_{\sigma(2)}) \in R^{\mathfrak{a}}$ for each $\sigma \in Sym(3)$ – and irreflexive – $\overline{a} \in R^{\mathfrak{a}} \Rightarrow \bigwedge_{i<j<3} a_i \neq a_j$). One then defines a relativized version,

$$d(-;\mathfrak{a}) : \mathcal{P}(\|\mathfrak{a}\|) \to \mathbb{N} : d(X;\mathfrak{a}) = \inf\{\delta(\mathfrak{a}[B]) : X \subseteq B \subseteq \|\mathfrak{a}\|\}$$

for each $\mathfrak{a}$, and then one defines a strong-substructure relation,

$$\mathfrak{a} \leq_s \mathfrak{b} \ \Leftrightarrow\ \left[\mathfrak{a} \leq \mathfrak{b} \ \&\ d(-;\mathfrak{a}) = d(-;\mathfrak{b}) \restriction \mathcal{P}(\|\mathfrak{a}\|)\right].$$

Classes of structures refining

$$\mathbf{C}_0 = \{\mathfrak{a} : \mathfrak{o} \leq_s \mathfrak{a}\}$$

where $\mathfrak{o} := \mathfrak{a}[\emptyset]$, organized by $\leq_s$ are the essential tools in all of the rest of the construction of a "pre-collapse" structure of Morley rank ω and a "post-collapse" strongly-minimal set that does not interpret an infinite field.

In [3], it is shown that strong-substructure relations defined from pre-dimensions generally need not produce model-complete structures, so there is certainly some gap between those sorts of formulations and our notion of a semi-coherent class. In this subsection, we will briefly discuss this more general context and a transformation that returns us to semi-coherent classes by changing language. Noting that model-completeness is not a language-invariant concept, it probably comes as no surprise that such a transformation is possible.

The presentation here is modeled loosely on those in [3] and small parts of [1].

Definition 2.9. Let $\mathbf{C}$ be a class of finite $\mathscr{L}$-structures, and let $\leq_s$ be a binary relation on $\mathbf{C}$ satisfying the following axioms:

ASC1: $\mathfrak{a} \leq_s \mathfrak{a}$ whenever $\mathfrak{a} \in \mathbf{C}$
ASC2: For all $\mathfrak{a}, \mathfrak{b} \in \mathbf{C}$, if $\mathfrak{a} \leq_s \mathfrak{b}$, then $\mathfrak{a} \leq \mathfrak{b}$.
ASC3: For all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbf{C}$, if $\mathfrak{a} \leq_s \mathfrak{b}$ and $\mathfrak{b} \leq_s \mathfrak{c}$, then $\mathfrak{a} \leq_s \mathfrak{c}$.

ASC4: For all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbf{C}$, if $\mathfrak{a} \leq_s \mathfrak{c}$, $\mathfrak{b} \leq_s \mathfrak{c}$, and $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{a} \leq_s \mathfrak{b}$.

ASC5: For all $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathbf{C}$, there are $\mathfrak{b} \in \mathbf{C}$ and embeddings $f_i : \mathfrak{a}_i \to \mathfrak{b}$ $(i < 2)$ such that $f[\mathfrak{a}_i] \leq_s \mathfrak{b}$.

(In all that follows, an embedding $f : \mathfrak{a} \to \mathfrak{b}$ such that $f[\mathfrak{a}] \leq_s \mathfrak{b}$ $(\mathfrak{a}, \mathfrak{b} \in \mathbf{C})$ will be called an *s-embedding*.)

ASC6: Given $\mathfrak{a}, \mathfrak{b}_0, \mathfrak{b}_1 \in \mathbf{C}$ and s-embeddings $f_i : \mathfrak{a} \to \mathfrak{b}_i$ $(i < 2)$, there are $\mathfrak{c} \in \mathbf{C}$ and s-embeddings $f_i' : \mathfrak{b}_i \to \mathfrak{c}$ such that $f_0' \circ f_0 = f_1' \circ f_1$.

Assuming $\leq_s$ satisfies these 6 axioms relative to $\mathbf{C}$, the pair $(\mathbf{C}, \leq_s)$ will be called an *abstract semi-coherent class* while $\leq_s$ itself is a strong-substructure relation.

An abstract semi-coherent class $(\mathbf{C}, \leq_s)$ is an *abstract coherent class* if there is a function $\lambda : \omega \to \omega$ such that for all $\mathfrak{a} \in \mathbf{C}$ and $X \subseteq \|\mathfrak{a}\|$, there are $\mathfrak{b}, \mathfrak{b}' \in \mathbf{C}$ such that $\mathfrak{a} \leq_s \mathfrak{b}'$, $\mathfrak{b} \leq_s \mathfrak{b}'$, $X \subseteq \|\mathfrak{b}\|$, and $|\mathfrak{b}| \leq \lambda(|X|)$.

The next two definitions allow us to make sense of "types" in a context where first-order formulas are not necessarily meaningful. As for AECs, we build a new notion of Galois type on just the idea of "identifiability under s-maps."

Definition 2.10 (s-maps, etc)**.** Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class. Let $\mathfrak{a}, \mathfrak{a}' \in \mathbf{C}$ and $B \subseteq \|\mathfrak{a}\|$, and let $f : B \to \|\mathfrak{a}'\|$ be some function. We say that f is a *partial s-embedding* – or more briefly, an *s-map* – if there are $\mathfrak{b} \in \mathbf{C}$ and an s-embedding $f_1 : \mathfrak{a} \to \mathfrak{b}$ such that $f \subseteq f_1$ and $\mathfrak{a}' \leq_s \mathfrak{b}$.

Given $\mathfrak{a} \in \mathbf{C}$ and a subset $B \subseteq \|\mathfrak{a}\|$, we write $\mathbf{C}_{(B;\mathfrak{a})}$ for the class pairs $(B; \mathfrak{b})$ in which $\mathfrak{b} \in \mathbf{C}$ is such that $B \subseteq \|\mathfrak{b}\|$ and $id_B : B \to \mathfrak{b}$ is an s-map. Obviously, $(B; \mathfrak{a}) \in \mathbf{C}_{(B;\mathfrak{a})}$.

For $\mathfrak{a} \in \mathbf{C}$, $B \subseteq \|\mathfrak{a}\|$, and $0 < n < \omega$, we define $\mathbf{C}^n_{(B;\mathfrak{a})}$ to be the set pairs $((B; \mathfrak{b}), \bar{b})$, where $(B; \mathfrak{b}) \in \mathbf{C}_{(B;\mathfrak{a})}$ and $\bar{b} \in \|\mathfrak{b}\|^n$.

Definition 2.11 (Galois types)**.** Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class. Let $\mathfrak{a} \in \mathbf{C}$, $B \subseteq \|\mathfrak{a}\|$, and $0 < n < \omega$. Then, for $((B; \mathfrak{b}_0), \bar{b}_0)$ and $((B; \mathfrak{b}_1), \bar{b}_1)$ in $\mathbf{C}^n_{(B;\mathfrak{a})}$, we assert

$$((B; \mathfrak{b}_0), \bar{b}_0) \equiv_{\mathrm{Gal}} ((B; \mathfrak{b}_1), \bar{b}_1)$$

if $id_B \cup \{(b_{0,i}, b_{1,i})\}_{i<n} : B\bar{b}_0 \to \mathfrak{b}_1$ and $id_B \cup \{(b_{1,i}, b_{0,i})\}_{i<n} : B\bar{b}_1 \to \mathfrak{b}_0$ are both s-maps. It's not hard to check that $\equiv_{\mathrm{Gal}}$ is an equivalence relation. We define,

$$\mathrm{tp}^{\mathfrak{b}}_{\mathrm{Gal}}(\bar{b}/B) = ((B; \mathfrak{b}), \bar{b})/\equiv_{\mathrm{Gal}}$$

which we call the Galois type of $\bar{b}$ over B in $\mathfrak{b}$. Finally, we define $S^{\mathrm{Gal}}_n(B; \mathfrak{a})$ to be the set of Galois types of n-tuples $\bar{b} \in \|\mathfrak{b}\|^n$ over B in $\mathfrak{b}$, where $(B; \mathfrak{b})$ ranges over $\mathbf{C}_{(B;\mathfrak{a})}$.

Observation 2.12. It's not hard to see that the axioms of an abstract semi-coherent class are sufficient to carry out Fraïssé's construction relative to $\leq_s$. Thus, if $(\mathbf{C}, \leq_s)$ is an abstract semi-coherent class, then the construction does produce a countably infinite $\mathscr{L}$-structure $\mathcal{M}$, which we call, for the moment, the *formal generic model of* $(\mathbf{C}, \leq_s)$. Once we've obtained $\mathcal{M}$, however, we need to specify how $\leq_s$ carries over to it, and for this we need one more definition.

Definition 2.13. Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class, and let $\mathcal{M}$ be its formal generic model.

- For $\mathfrak{a} \in \mathbf{C}$ such that $\mathfrak{a} \leq \mathcal{M}$, we define $\mathfrak{a} \leq_s \mathcal{M}$ to mean that for every $B \subset_{\mathrm{fin}} M$, there is a $\mathfrak{b} \in \mathbf{C}$ such that $B \subseteq \|\mathfrak{b}\|$ and $\mathfrak{a} \leq_s \mathfrak{b} \leq \mathcal{M}$. The notion of s-embedding is then modified in the obvious way.
- We define $\mathbf{C}^s(\mathcal{M})$ to be the set $\{\mathfrak{a} \in \mathbf{C} : \mathfrak{a} \leq_s \mathcal{M}\}$

In the following proposition, we find that the relationship between an abstract semi-coherent class $(\mathbf{C}, \leq_s)$ and its formal generic model is only slightly more complicated than that between a semi-coherent class and its generic model.

Proposition 2.14. *Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class, and let $\mathcal{M}$ be its formal generic model.*

I. $\mathcal{M}$ has the following properties:
  1. For every $B \subseteq M$, there is an $\mathfrak{a} \in \mathbf{C}^s(\mathcal{M})$ such that $B \subseteq \|\mathfrak{a}\|$.
  2. For every $\mathfrak{a} \in \mathbf{C}$, there is an s-embedding $\mathfrak{a} \to \mathcal{M}$.
  3. For all $\mathfrak{a}, \mathfrak{b} \in \mathbf{C}$ such that $\mathfrak{a} \leq_s \mathfrak{b}$ and any s-embedding $f_0 : \mathfrak{a} \to \mathcal{M}$, there is an s-embedding $f : \mathfrak{b} \to \mathcal{M}$ such that $f_0 \subseteq f$.

II. If $\mathcal{M}'$ is another outcome of Fraïssé construction for $(\mathbf{C}, \leq_s)$, then $\mathcal{M}' \cong \mathcal{M}$, and if $\mathcal{M}''$ is any countable structure satisfying the three sub-items of item I, then $\mathcal{M}'' \cong \mathcal{M}$.

III. Let $B \subset_{fin} M$, and let $\overline{a}, \overline{a}' \in M^n$ for some $0 < n < \omega$. The following are equivalent:
  1. $\mathrm{tp}^{\mathcal{M}}(\overline{a}/B) = \mathrm{tp}^{\mathcal{M}}(\overline{a}'/B)$.
  2. $\mathrm{tp}^{\mathfrak{b}}_{\mathrm{Gal}}(\overline{a}/B) = \mathrm{tp}^{\mathfrak{b}'}_{\mathrm{Gal}}(\overline{a}'/B)$ for some $\mathfrak{b}, \mathfrak{b}' \in \mathbf{C}^s(\mathcal{M})$ such that $B\overline{a} \subseteq \|\mathfrak{b}\|$ and $B\overline{a}' \subseteq \|\mathfrak{b}'\|$.
  3. For any $\mathfrak{b}, \mathfrak{b}' \in \mathbf{C}^s(\mathcal{M})$, if $B\overline{a} \subseteq \|\mathfrak{b}\|$ and $B\overline{a}' \subseteq \|\mathfrak{b}'\|$, then $\mathrm{tp}^{\mathfrak{b}}_{\mathrm{Gal}}(\overline{a}/B) = \mathrm{tp}^{\mathfrak{b}'}_{\mathrm{Gal}}(\overline{a}'/B)$.

Proof. Items I and II contain nothing especially novel beyond what is said for classes organized by the ordinary substructure relation (all of which are immediate observations from Fraïssé's construction itself). For III, it is enough to observe, firstly, that $\mathcal{M}$ is the prime model of its theory, so it is $\aleph_0$-homogeneous. Secondly, one observes (using AP relative to $\leq_s$) that complete types in $\mathcal{M}$ correspond precisely to Galois types of its finite strong-substructures □

The next corollary shows that abstract coherent classes are related to abstract semi-coherent classes in precisely the same way that coherent classes are related to semi-coherent classes.

Corollary 2.15. *Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class, and let $\mathcal{M}$ be its formal generic model. The following are equivalent:*

1. $\mathcal{M}$ is $\aleph_0$-categorical.
2. For every $0 < n < \omega$, there is a number k_n such that for every $\mathfrak{a} \in \mathbf{C}$, $\left|S^{\mathrm{Gal}}_n(\emptyset; \mathfrak{a})\right| \leq k_n$.
3. $\mathbf{C}$ is an abstract coherent class.

Proof. The equivalence of items 1 and 2 follows immediately from Proposition 2.14. For 1⇒3, one observes that, together with Proposition 2.14, the *lack* of a weak Löwenheim-Skolem function would yield a non-isolated type over $\emptyset$ for the theory of the formal generic model $\mathcal{M}$.

It remains only to prove 3⇒2. So, let $\lambda : \omega \to \omega$ be such that for all $\mathfrak{a} \in \mathbf{C}$ and $X \subseteq \|\mathfrak{a}\|$, there are $\mathfrak{b}, \mathfrak{b}' \in \mathbf{C}$ such that $\mathfrak{a} \leq_s \mathfrak{b}'$, $\mathfrak{b} \leq_s \mathfrak{b}'$, $X \subseteq \|\mathfrak{b}\|$, $|\mathfrak{b}| \leq \lambda(|X|)$. Given $0 < n < \omega$, we find that

$$\max\left\{\left|S^{\mathrm{Gal}}_n(\emptyset; \mathfrak{a})\right| : \mathfrak{a} \in \mathbf{C}\right\} \leq k_n := \left|\, \{(\mathfrak{b}, \overline{a})/\cong\, : \mathfrak{b} \in \mathbf{C},\, \overline{a} \in \|\mathfrak{b}\|^n,\, |\mathfrak{b}| \leq \lambda(n)\} \,\right|$$

where $(\mathfrak{b}, \overline{a})$, here, denotes a structure in the language $\mathscr{L}(\overline{c})$ expanding $\mathscr{L}$ by new constant symbols $c_0, ..., c_{n-1}$. □

We have seen that Galois types (over $\emptyset$) of an abstract semi-coherent class $(\mathbf{C}, \leq_s)$ correspond more or less exactly to complete types of the first-order theory of its formal generic model. We are about to see that by simply naming the Galois types as relation symbols, we can convert the abstract semi-coherent class $(\mathbf{C}, \leq_s)$ into a semi-coherent class $\mathbf{C}_s$ organized by the ordinary substructure relation.

Definition 2.16. Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class, and let $\mathcal{M}$ be its formal generic model.

Let $\mathscr{L}(\leq_s)$ be the language whose signature σ_s is constructed as follows: For each $0 < n < \omega$, each $\mathfrak{a} \in \mathbf{C}$, and each irreflexive $p \in S^{\mathrm{Gal}}_n(\emptyset; \mathfrak{a})$, σ_s has an n-ary relation symbol R_p. Given $\mathfrak{b} \in \mathbf{C}$ – which is an $\mathscr{L}$-structure – then we define $\mathfrak{b}_s$ to be the $\mathscr{L}(\leq_s)$-structure with universe $\|\mathfrak{b}\|$ and interpretations,

$$R_p^{\mathfrak{b}_s} = \left\{ \bar{b} \in \|\mathfrak{b}\|^n : \mathrm{tp}^{\mathfrak{b}}_{\mathrm{Gal}}(\bar{b}/\emptyset) = p \right\}.$$

Similarly, we define an $\mathscr{L}(\leq_s)$-structure $\mathcal{M}_s$ with universe M and interpretations,

$$R_p^{\mathcal{M}_s} = \left\{ \bar{b} \in M^n : (\exists \mathfrak{b} \in \mathbf{C}^s(\mathcal{M}))\ \mathrm{tp}^{\mathfrak{b}}_{\mathrm{Gal}}(\bar{b}/\emptyset) = p \right\}$$

Finally, we define $\mathbf{C}_s = \{\mathfrak{a}_s : \mathfrak{a} \in \mathbf{C}\}$.

With these definitions, we can formally state the transformation from abstract semi-coherent class to semi-coherent class. Because this transformation is available and in this article, we are mainly interested in language-invariant properties of classes of finite structures, we will have very little to say about abstract semi/coherent classes down the road.

Lemma 2.17. *Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class. Then $\mathbf{C}_s$ is a semi-coherent class of $\mathscr{L}(\leq_s)$-structures.*

Proof. For both JEP and AP, we note that the mapping $\mathbf{C} \to \mathbf{C}_s : \mathfrak{b} \mapsto \mathfrak{b}_s$ is an isomorphism of categories, where the morphisms in $\mathbf{C}$ are s-embeddings, and the morphisms of $\mathbf{C}_s$ are ordinary embeddings. That this *is* an isomorphism is a consequence of Proposition 2.14.III. □

Corollary 2.18. *Let $(\mathbf{C}, \leq_s)$ be an abstract semi-coherent class, and let $\mathcal{M}$ be its formal generic model. Also, let $\mathcal{N}$ be the generic model of the semi-coherent class $\mathbf{C}_s$. Then:*

(1) $\mathcal{M}_s \cong \mathcal{N}$
(2) If $f : \mathcal{M}_s \to \mathcal{N}$ is an isomorphism, then

$$\mathbf{C}^s(\mathcal{M}) = \{\mathfrak{a} \leq \mathcal{M} : f[\mathfrak{a}] \in \mathbf{C}_s(\mathcal{N})\}.$$

2.4. **Generic models via Baire category theory.** To conclude this Section 2, we remark briefly on the topological setting that makes the word "generic" appropriate for the structures recovered through Fraïssé's construction (Proposition 2.3). These considerations were/are implicit in essentially all of the literature on classes of finite structures with an amalgamation property. The present author has found the presentations in [20] and [16] to be especially illuminating.

We also use [20] as an outline for a topological approach to þ-forking in generic models of semi-coherent classes; this is presented in Section 5 below.

Definition 2.19. Let $\mathbf{C}$ be a class of finite $\mathscr{L}$-structures, and let M be a countably infinite set. Let $\mathbb{X}_{\mathbf{C}}$ be the set of all $\mathscr{L}$-structures $\mathcal{M}$ with universe M such that for every $B \subset_{\mathrm{fin}} M$, there is an $\mathfrak{a} \in \mathbf{C}$ such that $B \subseteq \|\mathfrak{a}\|$ and $\mathfrak{a} \leq \mathcal{M}$.

$\mathbb{X}_{\mathbf{C}}$ carries a topology with $\{[\mathfrak{a}] : \mathfrak{a} \in \mathbf{C}[M]\}$ as a base of clopen sets, where for each $\mathfrak{a} \in \mathbf{C}[M]$, $[\mathfrak{a}] = \{\mathcal{M} \in \mathbb{X}_{\mathbf{C}} : \mathfrak{a} \leq \mathcal{M}\}$. We also note that under this topology, the natural action

$$Sym(M) \times \mathbb{X}_{\mathbf{C}} \to \mathbb{X}_{\mathbf{C}} : (g, \mathcal{M}) \mapsto g\mathcal{M}$$

of $Sym(M)$ on $\mathbb{X}_{\mathbf{C}}$ is continuous, and for any $g \in Sym(M)$, $\mathcal{M} \in \mathbb{X}_{\mathbf{C}}$, it is clear that g is an isomorphism $\mathcal{M} \to g\mathcal{M}$. For brevity, we write $\overline{\mathcal{M}}$ for the $Sym(M)$-orbit of a structure $\mathcal{M} \in \mathbb{X}_{\mathbf{C}}$. We say that $\mathcal{M} \in \mathbb{X}_{\mathbf{C}}$ is *generic* (or generic for $\mathbf{C}$) if $\overline{\mathcal{M}}$ is a co-meager subset of $\mathbb{X}_{\mathbf{C}}$.

Theorem 2.20. *Let $\mathbf{C}$ be a class of finite structures. If $\mathbf{C}$ is semi-coherent, then some orbit $\overline{\mathcal{M}}$ is co-meager, and $\mathcal{M}$ is the generic model of $\mathbf{C}$ in the sense of Fraïssé's construction.*

3. Adherence of definable sets in semi/coherent classes

In this section, we nail down some notions of "adherence" of definable sets in a semi-coherent class modeled on the classical situation induced by the elementary submodel relation. Our point of departure will be the notion of a *robust chain* defined and examined in [25]. Since those authors work with a fixed chain of finite structures, while we prefer to work with semi-coherent classes, their definition is not immediately applicable. We use a straightforward notion of foundation rank in a class **C** (see Definition 3.2) to provide the grading that is provided by the chain in [25]. Working relative to the foundation rank, we define two notions of robustness for semi-coherent classes, robustness and super-robustness. The former is a reasonable approximation to the scenario of a robust chain, and the latter is roughly analogous to the scenario of a robust chain of *chain-complexity 0*. The definition of super-robustness finally puts us in a position to define the two kinds of classes of finite structures that gave their names to this article.

Definition 3.1. Let **C** be a class of finite structures.

- We say that **C** is *tame* if it is coherent and is cofinal with a super-robust class.
- We say that **C** is *semi-tame* if it is semi-coherent and is cofinal with a super-robust class.

It seems that in practice, it is always fairly natural to assume that **C** is initially as large as possible – replacing **C** with the union of *all* semi-coherent classes that are cofinal with **C** causes no difficulties. Consequently, little or nothing changes if we alter the definitions of tame and semi-tame to read, "...contains a cofinal super-robust sub-class," and we often switch back and forth between these points of view.

With the basic definitions in place, we define several additional conditions that a semi-coherent might possess – a well-quasi-ordering condition, two ways of interacting very nicely with ultraproducts, and a strengthened Löwenheim-Skolem condition – and we state characterization theorems (Theorem 3.8 and Corollary 3.10) of super-robustness *relative* to robustness; the proofs of these theorems are long and reasonably basic, so they are banished to Appendix A below. Initially, it will seem unlikely that one would prove that a class **C** is super-robust by first showing that it is robust and then proving one of the more esoteric conditions equivalent to super-robustness. However, we believe that each of these conditions has a role to play in other work. For example:

- The well-quasi-ordering condition is useful in analyzing quasi-finite axiomatizability of certain kinds of $\aleph_0$-categorical theories. (See [13].)
- The ultraproduct condition called perfect-FSP is useful in analyzing certain kinds of 0,1-law. Such an analysis is carried out in Section 6 of this paper.

We might also note that the other ultraproduct condition, generic-categoricity, is, in a sense, the natural naive translation of 0,1-laws into a context that eschews probability theory. (It is also deployed in [15].)

Finally, we would be remiss if we did not mention that our notion of super-robustness has close affinities with the graded notions of saturation identified and exploited in [21] as a sufficient condition for the finite sub-model property. Indeed, super-robustness can be seen as little more than a formalization of "graded saturation" within a class of finite structures. In this light, the condition called perfect-FSP is, again, a natural point for discussion.

As previously mentioned, our definitions around adherence of definable sets are built atop the notion of foundation rank in a class **C** of finite structures. We give that definition now, and immediately thereafter, we present the relevant versions of robustness of a semi-coherent class **C**.

Definition 3.2. Let $\mathbf{C}$ be a class of finite structures. We define its foundation rank, $\mathrm{rk}^{\mathbf{C}}$, as follows:

$$\begin{cases} \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq 0 & \text{for all } \mathfrak{a} \in \mathbf{C} \\ \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq n+1 & \text{if } \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}_0) \geq n \text{ for some } \mathfrak{a}_0 \lneq \mathfrak{a} \text{ such that } \mathfrak{a}_0 \in \mathbf{C}. \end{cases}$$

When $\mathbf{C}$ is clear from context (as it usually will be), we write rk instead of $\mathrm{rk}^{\mathbf{C}}$. We also define a family of relative foundation ranks; for each $\mathfrak{a} \in \mathbf{C}$, $\mathrm{rk}^{\mathbf{C}}(-/\mathfrak{a}) = \mathrm{rk}^{\mathbf{C}_{\mathfrak{a}}}$, where

$$\mathbf{C}_{\mathfrak{a}} = \{\mathfrak{b} \in \mathbf{C} : \mathfrak{a} \leq \mathfrak{b}\}, \text{ and (if } \mathfrak{a} \in \mathbf{C}(\mathcal{M})) \;\; \mathbf{C}_{\mathfrak{a}}(\mathcal{M}) = \{\mathfrak{b} \in \mathbf{C}(\mathcal{M}) : \mathfrak{a} \leq \mathfrak{b}\}.$$

Definition 3.3. Let $\mathbf{C}$ be a semi-coherent class with generic model $\mathcal{M}$.

Below are three possible definitions of robustness, all expressing to some degree that definable sets of the generic model "make sense" in its finite approximating structures – i.e. members of $\mathbf{C}$. The first definition, of robust-chainable classes, is essentially identical to the definition given in [25], and we include it primarily for the sake of comparison.[1] The second definition, of robust classes, is what we think of as the nearest purely-rank-based approximation to robust-chainability that one can hope for in the context of semi-coherent classes. Finally, we have super-robustness, which is the strongest refinement of robustness that, to us, still seems basically reasonable.[2]

Robust-chainable: We say that $\mathbf{C}$ is *robust-chainable* if there are a sequence $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ and a function $\nu : \mathscr{L} \times \omega \to \omega$ such that:

i. For all $m < n < \omega$, $\mathfrak{a}_m \leq \mathfrak{a}_n$, and $\bigcup_n \mathfrak{a}_n = \mathcal{M}$.

ii. For all $\varphi(x_0, ..., x_{k-1}) \in \mathscr{L}$, $m \leq n < p < \omega$, and $\bar{b} \in \|\mathfrak{a}_m\|^k$,

$$\nu(\varphi, m) \leq n \;\Rightarrow\; \big(\mathfrak{a}_n \models \varphi(\bar{b}) \;\Leftrightarrow\; \mathfrak{a}_p \models \varphi(\bar{b})\big).$$

Robust: We say that $\mathbf{C}$ is *robust* if there is function $\nu : \mathscr{L} \times \omega \to \omega$ such that for all $\varphi(x_0, ..., x_{k-1}) \in \mathscr{L}$, $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2 \in \mathbf{C}$, and $\bar{b} \in \|\mathfrak{a}\|^k$, if $\mathfrak{a}_0 \leq \mathfrak{a}_1 \leq \mathfrak{a}_2$ and

$$\mathrm{rk}(\mathfrak{a}_1/\mathfrak{a}_0) \geq \nu(\varphi, \mathrm{rk}(\mathfrak{a}_0))$$

then

$$\mathfrak{a}_1 \models \varphi(\bar{b}) \;\Leftrightarrow\; \mathfrak{a}_2 \models \varphi(\bar{b}).$$

Super-robust: We say that $\mathbf{C}$ is *super-robust* if there is function $\nu : \mathscr{L} \to \omega$ such that for all $\varphi(x_0, ..., x_{k-1}) \in \mathscr{L}$, $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathbf{C}$, and $\bar{b} \in \|\mathfrak{a}\|^k$, if $\mathfrak{a}_0 \leq \mathfrak{a}_1$ and

$$\mathrm{rk}(\mathfrak{a}_0) \geq \nu(\varphi)$$

then

$$\mathfrak{a}_0 \models \varphi(\bar{b}) \;\Leftrightarrow\; \mathfrak{a}_1 \models \varphi(\bar{b}).$$

We now turn to the definitions of some more esoteric conditions – Definitions 3.4, 3.5, 3.6, and 3.7 – that will turn out to be, *modulo* robustness, equivalent to super-robustness.

Definition 3.4. Let $\mathbf{C}$ be a semi-coherent class.

- For clarity, we define a relation $\leq^*$ on $\mathbf{C}$ as follow: for $\mathfrak{a}, \mathfrak{b} \in \mathbf{C}$, $\mathfrak{a} \leq^* \mathfrak{b}$ if there is an embedding $\mathfrak{a} \to \mathfrak{b}$.
- We say that $\mathbf{C}$ is *well-quasi-ordered by $\leq^*$ up to foundation rank* if for any sequence $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ of members of $\mathbf{C}$, if the set $\{\mathrm{rk}(\mathfrak{a}_n) : n < \omega\}$ is infinite, then there are $m < n < \omega$ such that $\mathfrak{a}_m \leq^* \mathfrak{a}_n$.
- We say that $\mathbf{C}$ is *well-quasi-ordered by $\leq^*$* if for any sequence $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ of members of $\mathbf{C}$, there are $m < n < \omega$ such that $\mathfrak{a}_m \leq^* \mathfrak{a}_n$.

[1] It is also true that if an axiomatic strong substructure relation is permitted, one can obtain their definition pretty much exactly.

[2] We should also acknowledge that our notion of super-robustness actually grew out of an initial misreading of the definition of a robust chain as presented in [25] – we had the quantifiers all wrong.

Definition 3.5. Let $\mathbf{C}$ be a semi-coherent class. We say that some function $\lambda : \omega \to \omega$ is a *Löwenheim-Skolem function (LS-function) for* $\mathbf{C}$ if for any $\mathfrak{a} \in \mathbf{C}$ and $X \subseteq \|\mathfrak{a}\|$, there is a substructure $\mathfrak{a}_0 \leq \mathfrak{a}$ such that $X \subseteq \|\mathfrak{a}_0\|$, $\mathfrak{a}_0 \in \mathbf{C}$, and $|\mathfrak{a}_0| \leq \lambda(|X|)$.

Definition 3.6. Let $\mathbf{C}$ be a semi-coherent class. We say that $\mathbf{C}$ is *generically-categorical up to foundation rank* if for every enumeration of $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ of the isomorphism types of $\mathbf{C}$, for any non-principal ultrafilter $\mathscr{U}$ on ω, if

$$\{n : \mathrm{rk}(\mathfrak{a}_n) \geq e\} \in \mathscr{U}$$

for every $e < \omega$, then $\prod_n \mathfrak{a}_n/\mathscr{U} \vDash T_{\mathbf{C}}$.

Similarly, we say that $\mathbf{C}$ is simply *generically-categorical* if for every enumeration of $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ of the isomorphism types of $\mathbf{C}$, for any non-principal ultrafilter $\mathscr{U}$ on ω, $\prod_n \mathfrak{a}_n/\mathscr{U} \vDash T_{\mathbf{C}}$.

Definition 3.7. Let $\mathbf{C}$ be a semi-coherent class.

- For the sake of less cumbersome notation, we write $\Pi\mathbf{C}(\mathcal{M})$ in place of the usual notation $\prod_{\mathfrak{a}\in\mathbf{C}(\mathcal{M})} \mathfrak{a}$ for direct products. Thus, speaking literally, $\Pi\mathbf{C}(\mathcal{M})$ is the set of functions $f : \mathbf{C}(\mathcal{M}) \to M$ such that $f(\mathfrak{b}) \in \|\mathfrak{b}\|$ for every $\mathfrak{b} \in \mathbf{C}(\mathcal{M})$.
- We define the family $\mathsf{Cone}_{\mathbf{C}}(\mathcal{M})$ to be,

  $$\{X \subseteq \mathbf{C}(\mathcal{M}) : (\exists \mathfrak{a} \in \mathbf{C}(\mathcal{M}))\, \mathbf{C}_{\mathfrak{a}}(\mathcal{M}) \subseteq X\}$$

  which is a proper filter on $\mathbf{C}(\mathcal{M})$.
- Similarly, we define the family $\mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathcal{M})$ to be,

  $$\{X \subseteq \mathbf{C}(\mathcal{M}) : (\exists \mathfrak{a} \in \mathbf{C}(\mathcal{M})(\forall r < \omega)\, \mathbf{C}_{\mathfrak{a}}(\mathcal{M}) \setminus Z_r \subseteq X\}$$

  where for each $r < \omega$,

  $$Z_r = \left\{\mathfrak{a} \in \mathbf{C}(\mathcal{M}) : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \leq r\right\}.$$
- We fix a function $w : M \to \Pi\mathbf{C}(\mathcal{M}) : a \mapsto w_a$ such that for all $\mathfrak{b} \in \mathbf{C}(\mathcal{M})$, if $a \in \|\mathfrak{b}\|$, then $w_a(\mathfrak{b}) = a$.
- We will say that $\mathbf{C}$ has the *perfect Finite Sub-model Property (perfect-FSP)* if for any ultrafilter $\mathscr{U}$ on $\mathbf{C}(\mathcal{M})$ such that $\mathsf{Cone}_{\mathbf{C}}(\mathcal{M}) \subseteq \mathscr{U}$, the map

  $$w^{\mathscr{U}} : M \to \Pi\mathbf{C}(\mathcal{M})/\mathscr{U} : a \mapsto w_a/\mathscr{U}$$

  is an elementary embedding $\mathcal{M} \to \Pi\mathbf{C}(\mathcal{M})/\mathscr{U}$.
- We will say that $\mathbf{C}$ has the *perfect Finite Sub-model Property up to foundation rank (perfect-FSP/rank)* if for any ultrafilter $\mathscr{U}$ on $\mathbf{C}(\mathcal{M})$ such that $\mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathcal{M}) \subseteq \mathscr{U}$, $w^{\mathscr{U}} : \mathcal{M} \to \Pi\mathbf{C}(\mathcal{M})/\mathscr{U}$ is an elementary embedding.

We now state the theorem characterizing super-robustness in several ways *modulo* robustness for a semi-coherent class. Thereafter, we state a related result characterizing super-robustness for coherent classes. As mentioned earlier, the demonstrations of these facts may be found in Appendix A.

Theorem 3.8. *Let* $\mathbf{C}$ *be a robust semi-coherent class. The following are equivalent:*

(1) $\mathbf{C}$ *is super-robust.*
(2) For every $\mathfrak{a} \in \mathbf{C}$*,* $\mathbf{C}_{\mathfrak{a}}$ *is well-quasi-ordered by* $\leq^*$ *up to foundation rank.*
(3) $\mathbf{C}$ *is generically-categorical up to foundation rank.*
(4) $\mathbf{C}$ *has perfect-FSP/rank.*

Observation 3.9. If $\mathbf{C}$ is such that for every $e < \omega$, $\{\mathfrak{a}/\cong : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \leq e\}$ is finite, then in every statement of the form "... up to foundation rank," that part of the statement is always true of the object in question, so can be removed from the formulation. Thus, items 1-4 of Corollary 3.10 are seen to be equivalent to each other without any additional effort.

Corollary 3.10. *Let $\mathbf{C}$ be a robust* <u>*coherent*</u> *class. Then, the following are equivalent:*

(1) $\mathbf{C}$ *is super-robust.*
(2) For every $\mathfrak{a} \in \mathbf{C}$, $\mathbf{C}_{\mathfrak{a}}$ is well-quasi-ordered by $\leq^*$.
(3) $\mathbf{C}$ *is generically-categorical.*
(4) $\mathbf{C}$ *has perfect-FSP.*
(5) $\mathbf{C}$ *possesses a Löwenheim-Skolem function.*

To conclude this section (which would otherwise have no proofs in it), we state and prove two facts that are directly relevant to the analysis of 0,1-laws carried out in Section 6 below.

Proposition 3.11. *Let $\mathbf{C}$ be a semi-coherent class. If $\mathbf{C}$ has perfect-FSP and is well-quasi-ordered by $\leq^*$ up to foundation rank, then there is a function $\mu : \mathscr{L} \times \mathbf{C}/\cong \to \omega$ such that for all $\varphi(x_0, ..., x_{k-1}) \in \mathscr{L}$, $\mathfrak{a} \in \mathbf{C}, \mathfrak{b}, \mathfrak{c} \in \mathbf{C}_{\mathfrak{a}}$, and $\bar{b} \in \|\mathfrak{a}\|^k$, if $\mathfrak{a} \leq \mathfrak{b} \leq \mathfrak{c}$ and*

$$\mathrm{rk}^{\mathbf{C}}(\mathfrak{b}/\mathfrak{a}) \geq \mu(\varphi, \mathfrak{a}/\cong)$$

then

$$\mathfrak{b} \models \varphi(\bar{b}) \iff \mathfrak{c} \models \varphi(\bar{b}).$$

From this, it follows that $\mathbf{C}$ has a robust – hence super-robust by Theorem 3.8 – cofinal sub-class.

Proof. Towards a contradiction, suppose no such function exists; in particular, we have a formula $\varphi(x_0, ..., x_{n-1}) \in \mathscr{L}$ and an isomorphism type $\mathfrak{a}/\cong$ for some $\mathfrak{a} \in \mathbf{C}$ witnessing the failure of the conclusion of the proposition. We assume that $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$, where $\mathcal{M}$ is the generic model of $\mathbf{C}$. Then, using the assumption that $\mathbf{C}$ is well-quasi-ordered by $\leq^*$ up to foundation rank, we see the following:

Observation. We may choose $X, Y \subseteq \mathbf{C}_{\mathfrak{a}}(\mathcal{M})$ and $\bar{a} \in \|\mathfrak{a}\|^n$ such that

- $X \cap \mathbf{C}_{\mathfrak{b}}(\mathcal{M}) \neq \emptyset$ and $Y \cap \mathbf{C}_{\mathfrak{b}}(\mathcal{M}) \neq \emptyset$ for all $\mathfrak{b} \in \mathbf{C}(\mathcal{M})$.
- For all $\mathfrak{b} \in X$, $\mathfrak{b} \models \varphi(\bar{a})$.
- For all $\mathfrak{c} \in Y$, $\mathfrak{c} \models \neg\varphi(\bar{a})$.

Extending $\{X\}$ and $\{Y\}$ to ultrafilters $\mathscr{U}_X$ and $\mathscr{U}_Y$ on $\mathbf{C}(\mathcal{M})$ containing $\mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathcal{M})$, we see that $\mathcal{M}$ fails to be an elementary submodel of at least one of $\Pi\mathbf{C}(\mathcal{M})/\mathscr{U}_X$ or $\Pi\mathbf{C}(\mathcal{M})/\mathscr{U}_Y$ – a contradiction to perfect-FSP. $\square$

Corollary 3.12. *Let $\mathbf{C}$ be a coherent class (so that for every $e < \omega$, $\{\mathfrak{a} \in \mathbf{C} : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \leq e\}$ is finite up to isomorphism). If $\mathbf{C}$ has perfect-FSP and is well-quasi-ordered by $\leq^*$, then $\mathbf{C}$ is tame.*

4. Examples and preservation properties

We have established several characterizing properties of tame and semi-tame classes of finite structures, but now we take a step to the side and consider several classes of examples that can be verified without appealing to these more esoteric properties. In Section 6, however, we use some of these considerations to generate a large family of [semi-]tame classes from consideration of 0,1-laws for first-order logic.[3]

[3]Here and in the sequel, we often use the formulation "[semi-]tame" as an abbreviation in statements like "If $\mathbf{C}$ is tame (respectively, semi-tame) and ..., then $\mathbf{C}'$ is tame (respectively, semi-tame)."

First of all, we will see that a certain family of strongly-minimal theories (that includes all $\aleph_0$-categorical strongly-minimal theories in countable languages) are always understandable as generic theories of semi-tame classes, and building on this fact, we will find that totally-categorical theories with a certain "bounded algebraic arity" property are also generic theories of tame classes. (The present author has referred to a condition called "small algebraicity" in the past; however it turns out that [24] addressed the some property under the name "bounded algebraic arity," which we feel now is more descriptive and euphonious.) To some degree, these results can be recovered from profound geometric results due to [6] and [34], but here, we choose to verify these examples briefly and from first principles, building on a lemma drawn from [30].

We should also observe that the generalization of [6, 34] to smoothly approximable structures in [7] yields that the class of finite structures isomorphic to the finite envelopes of a Lie coordinatizable structure form a tame class as well. However, the smoothly approximable structures do not include the random graph, so tameness is, to that extent, a proper generalization of smooth approximability.

After addressing strongly-minimal and totally-categorical theories, we turn to some preservation properties that allow one to construct new [semi-]tame classes from old. Specifically, we show that certain "reasonable" definable quotients of the definable sets in the generic model of a [semi-]tame class are themselves generic models of [semi-]tame classes. Thereafter, we consider finite direct products of and finite direct sums of finite structures (which are important constructions in finite model theory – e.g. see [22, 26]), showing that these constructions also preserve [semi-]tameness.

In the first three subsections that follow, we often assume that certain sets and equivalence relations are 0-definable. This is purely a matter of convenience; our development applies to objects defined with parameters as soon as one makes the following observation.

Observation 4.1. Let $\mathbf{C}$ be a [semi-]tame class with generic model $\mathcal{M}$. For every $B \subset_{\text{fin}} M$, $\mathbf{C}_B$ is [semi-]tame.

Throughout this section, our discussion is held "up to change of language," as outlined in Subsection 2.3 on abstract semi-coherent classes.

4.1. **Some strongly-minimal theories.** In this subsection, we consider certain strongly-minimal theories, showing that these theories (which include all $\aleph_0$-categorical strongly-minimal theories) arise as generic theories of [semi-]tame classes. Except for some adjustments to the notation and emphasis, the material in this subsection is really due to [30].

One important insight from [30] is that for strongly-minimal theories, certain "weakly-algebraically-closed" subsets are also "weakly-elementary substructures"; the weakness here comes from restricting attention to a finite set of formulas as in the following definition.

Definition 4.2. Let $\mathscr{L}$ be a some first-order language, and let $F \subset_{\text{fin}} \mathscr{L}$. For an $\mathscr{L}$-structure $\mathcal{M}$ and a subset $B \subseteq M$, we define

$$\mathrm{Acl}_F^{\mathcal{M}}(B) = \left\{ B' \subseteq M : \begin{array}{c} B \subseteq B', \\ \forall\, \varphi(x, \overline{y}) \in F, \overline{b} \in B' : \\ \quad\quad \left|\varphi(\mathcal{M}, \overline{b})\right| < \aleph_0 \;\Rightarrow\; \varphi(\mathcal{M}, \overline{b}) \subseteq B' \end{array} \right\}$$

and

$$\mathrm{acl}_F^{\mathcal{M}}(B) = \bigcap \mathrm{Acl}_F^{\mathcal{M}}(B).$$

Clearly, $\mathrm{acl}_F^{\mathcal{M}}(B) \subseteq \mathrm{acl}(B)$.

We lift the following lemma directly from [30], and we include its proof just for the sake of clarity. It demonstrates formally the insight that we mentioned above.

Lemma 4.3 ([30]). *Let $\mathcal{M}$ be a strongly-minimal $\mathscr{L}$-structure. There is a function $\kappa : \mathcal{P}_{fin}(\mathscr{L}) \to \omega$ such that for any $B \subseteq M$ and $F \subset_{fin} \mathscr{L}$, if $|B| \geq \kappa(F)$ and $\mathrm{acl}_F^{\mathcal{M}}(B) = B$, then $\mathcal{M}[B] \preceq_F \mathcal{M}$.*

Proof. Given $F \subset_{\text{fin}} \mathscr{L}$, by possibly passing to a finite extension $F \subseteq F' \subset_{\text{fin}} \mathscr{L}$, we assume that F is closed under taking sub-formulas. We define a function $f : F^p \to \omega$ as follows, where F^p denotes the set of partitioned formulas $\varphi(x, \overline{y})$ derived from members of $\mathscr{L}$:

Let $\varphi(x, \overline{y}) \in F^p$ be given. By strong-minimality, there is a number $n < \omega$ such that for all $\mathcal{N} \equiv \mathcal{M}$ and $\overline{b} \in N^{|\overline{y}|}$,

$$\varphi(\mathcal{N}, \overline{b}) \text{ is infinite } \Leftrightarrow \mathcal{N} \models \exists_{\geq n} x\, \varphi(x, \overline{b}).$$

Taking n_φ to be the smallest such number, we set $f(\varphi) = \max\{n_\varphi, n_{\neg\varphi}\}$, where $n_{\neg\varphi}$ is defined for $\neg\varphi(x, \overline{y})$ in a similar way.

We then define $\kappa(F) = \max\{f(\varphi) + 1 : \varphi(x, \overline{y}) \in F^p\}$. Of course, now we'll need to show that this choice works as advertised.

So, let $B \subseteq M$ be such that $|B| \geq \kappa(F)$ and $\mathrm{acl}_F^{\mathcal{M}}(B) = B$. By the Tarski-Vaught test restricted to F, it is enough to show that if $\varphi(x, \overline{y}) \in F^p$ and $\overline{b} \in B^{|\overline{y}|}$, then $\mathcal{M} \models \exists x\, \varphi(x, \overline{b})$ if and only if $\mathcal{M}[B] \models \exists x\, \varphi(x, \overline{b})$. By strong-minimality, either $\varphi(\mathcal{M}, \overline{b})$ is finite or $M \setminus \varphi(\mathcal{M}, \overline{b})$ is finite. If the former, then $\varphi(\mathcal{M}, \overline{b}) \subseteq B$ because $B = \mathrm{acl}_F^{\mathcal{M}}(B)$. If the latter, then since $|B| > f(\varphi) \geq n_{\neg\varphi}$, there is an element $c \in B$ such that $\mathcal{M} \models \neg\neg\varphi(c, \overline{b})$. Since F is closed under taking sub-formulas, we may say inductively, $\mathcal{M} \models \varphi(c, \overline{b})$ if and only if $\mathcal{M}[B] \models \varphi(c, \overline{b})$. We have therefore shown that $\mathcal{M} \models \exists x\, \varphi(x, \overline{b})$ if and only if $\mathcal{M}[B] \models \exists x\, \varphi(x, \overline{b})$, as required. ☐

Since $\mathrm{acl}_F^{\mathcal{M}}(B) \subseteq \mathrm{acl}(B)$ for any finite set of formulas F, the following corollary is immediate. Moreover, if one restricts attention to strongly-minimal theories such that, at least in the prime model, every finite set extends to an algebraically-closed finite set, Theorem 4.5 is also an easy consequence.

Corollary 4.4. *Let $\mathcal{M}$ be a strongly-minimal $\mathscr{L}$-structure. There is a function $\kappa : \mathcal{P}_{fin}(\mathscr{L}) \to \omega$ such that for any*

$$B \in \mathrm{acl}[\mathcal{M}] := \left\{C \subset_{fin} M : \mathrm{acl}^{\mathcal{M}}(C) = C\right\}$$

and any $F \subset_{fin} \mathscr{L}$, if $|B| \geq \kappa(F)$, then $\mathcal{M}[B] \prec_F \mathcal{M}$.

Theorem 4.5. *Let T be a strongly-minimal $\mathscr{L}$-theory with prime model $\mathcal{M}$, and suppose that $\mathcal{M}$ is locally finite (not necessarily* uniformly *locally finite) in the sense that for every $B \subset_{fin} M$, $\mathrm{acl}^{\mathcal{M}}(B)$ is finite. Let $\mathbf{C}^{\mathcal{M}}$ be the class of finite $\mathscr{L}$-structures obtained by closing $\{\mathcal{M}[B] : B \in \mathrm{acl}[\mathcal{M}]\}$ under isomorphisms. Then $\mathbf{C}^{\mathcal{M}}$ is a semi-tame class with $Th(\mathcal{M})$ as its generic theory.*

In particular, if $\mathcal{M}$ is a strongly-minimal $\mathscr{L}$-structure that is also $\aleph_0$-categorical, then $\mathbf{C}^{\mathcal{M}}$ is a tame class with $Th(\mathcal{M})$ as its generic theory; in fact, $\mathbf{C}^{\mathcal{M}}$ is itself super-robust.

4.2. **Some totally-categorical theories.** In this subsection, we sketch a proof that certain totally-categorical theories can be understood as generic theories of tame classes. These are the totally-categorical theories that have the following "bounded algebraic arity" property. It appears that it is true *all* totally-categorical theories have bounded algebraic arity, but we do not, at present, know how to prove this without appealing to the "geometric" technology of [6, 34]; hence, we leave this as a hypothesis for the moment.

Definition 4.6. We say that a theory T has *bounded algebraic arity* if there is a number $d < \omega$ such that the following holds:

For any $\mathcal{N} \models T$, $C \subseteq N$, and $a, b \in N$, if $a \in \mathrm{acl}^{\mathcal{N}}(bC) \setminus \mathrm{acl}^{\mathcal{N}}(C)$, then there is a subset $C_0 \subseteq \mathrm{acl}^{\mathcal{N}}(C)$ such that $|C_0| \leq d$, $a \in \mathrm{acl}^{\mathcal{N}}(bC_0)$, and $\left|a^{Aut(N/bC_0)}\right| = \left|a^{Aut(\mathcal{N}/bC)}\right|$.

Theorem 4.7. *Let T be a totally-categorical theory in a countable language $\mathscr{L}$. If T has bounded algebraic arity, then T is (up to a change of language) the generic theory of a tame class.*

Proof (sketch). Let $\mathcal{M}$ be countably infinite $\mathscr{L}$-structure such that $T = Th(\mathcal{M})$ is totally-categorical, and let $X \subseteq M$ be 0-definable strongly-minimal set such that $\mathcal{M}$ is prime over X. We write $\mathcal{M}_X$ for the structure with universe X equipped with every irreflexive 0-definable relation $R \subseteq X^n$ $(0 < n < \omega)$ from $\mathcal{M}$, writing $\mathscr{L}_X$ for this language. Finally, we assume that T has bounded algebraic arity.

Observation. Some observations that follow immediately from the hypotheses.

- There is a function $h : \omega \to \omega$ such that for every $0 < n < \omega$, for every $p \in S_n^{\mathcal{M}}(X)$, there is a set
$$C \in \mathrm{acl}[X; \mathcal{M}] := \left\{X \cap \mathrm{acl}^{\mathcal{M}}(B) : B \subset_{\mathrm{fin}} X\right\}$$
such that $p{\restriction}C \vDash p$ and $|C| \leq h(n)$.
- For every n, we may choose a finite set of formulas $F_n \subset_{\mathrm{fin}} \mathscr{L}$ such that (i) every $n{+}h(n)$-type of T is isolated by a formula in F_n, and (ii) every member of F_n is principal. Furthermore, we may fix a number $n_0 < \omega$ such if $n_0 \leq n$, then X is definable by a formula in F_n.
- Since T has with bounded algebraic arity, then there is a number $n_1 < \omega$ such that for any $n < \omega$, $\mathcal{N} \vDash T$, and $C \subseteq N$, if $n_1 \leq n$, then $\mathrm{acl}_{F_n}^{\mathcal{N}}(C) = \mathrm{acl}^{\mathcal{N}}(C)$.

Let F_n^X be the subset of F_n consisting of those $\varphi(x_0, ..., x_{k-1}) \in F_n$ such that $\varphi \vDash \bigwedge_{i<k} X(x_i)$ *modulo* T. Let P_n denote the set of partitioned formulas $\varphi(x, \overline{y})$ obtained by partitioning off one free variable from among those of a member of F_n. By $\aleph_0$-categoricity, for each $\varphi = \varphi(x, \overline{y})$, there is a number k such that
$$\varphi(\mathcal{N}, \overline{b}) \text{ is infinite } \Leftrightarrow \mathcal{N} \vDash \exists_{\geq k} x\, \varphi(x, \overline{b}).$$
whenever $\mathcal{N} \equiv \mathcal{M}$, and we take r_φ to be the smallest such number. Now, for $n \geq n_0 + n_1$ and $k = 0, 1, ..., n$, we define notions of (n, k)-good subsets of M.

0. $C \subset_{\mathrm{fin}} M$ is $(n, 0)$-good if:
   - $\mathrm{acl}^{\mathcal{M}}(C) = C$
   - $|C \cap X| \geq \kappa(F_n^X)$, where κ is given to us by Corollary 4.4.
   - For all $\overline{a} \in C^{\leq n}$, $\mathrm{tp}(\overline{a}/C \cap X) \vDash \mathrm{tp}(\overline{a}/X)$.

k+1. $C \subset_{\mathrm{fin}} M$ is $(n, k+1)$-good if:
   - C is (n, k)-good.
   - For all $\varphi(x; y_0, ..., y_{k-1}, z_0, ..., z_{\ell-1}) \in P_n$ such that
$$T \vDash \varphi(x, \overline{y}, \overline{z}) \to \bigwedge\nolimits_{i<k} \neg X(y_i) \wedge \bigwedge\nolimits_{j<\ell} X(z_j),$$
for all $\overline{b} \in (C \setminus X)^k$ and $\overline{c} \in (C \cap X)^\ell$, if $\varphi(\mathcal{M}, \overline{b}, \overline{c})$ is infinite, then
$$\left|C \cap \varphi(\mathcal{M}, \overline{b}, \overline{c})\right| \geq (n+1) \cdot (r_\varphi + 1).$$

If $C \in \mathrm{acl}[\mathcal{M}]$ is (n, k)-good, we will also say that C is an (n, k)-good continuation of the set $C \cap X$ and of the $\mathscr{L}_X$-structure $\mathcal{M}_X[C \cap X]$.

Observation. Let $k \leq n < \omega$, where $n \geq n_0 + n_1$. "Pulling down" an interpretation in $\mathcal{M}_X$, there are $N_{n,k} < \omega$ and an interpreting scheme $\Gamma_{n,k} \subset_{\mathrm{fin}} \mathscr{L}_X$ such that for all $\mathfrak{a} \in \mathbf{C}^{\mathcal{M}_X}$, if $\mathrm{rk}(\mathfrak{a}) \geq N_{n,k}$, then $\Gamma_{n,k}(\mathfrak{a})$ is an (n, k)-good continuation of $\mathfrak{a}$. It follows that for any $C_0 \in \mathrm{acl}[X; \mathcal{M}]$, if C_0 is sufficiently large, then there is an (n, k)-good continuation $C \in \mathrm{acl}[\mathcal{M}]$ of C_0.

For each $n_0 + n_1 \leq n < \omega$, let
$$\mathbf{C}_n = \left\{\mathcal{M}[C] : \begin{array}{l} C \in \mathrm{acl}[\mathcal{M}] \text{ is an } (n,n)\text{-good} \\ \text{continuation of } C \cap X \end{array}\right\}$$

up to isomorphism. We choose a sufficiently-fast-growing strictly increasing function $f : \omega \to \omega$, and we define

$$\mathbf{C}^* = \left\{\mathfrak{a} \in \bigcup_n \mathbf{C}_n : |\mathfrak{a}| \geq f(n) \Rightarrow \mathfrak{a} \in \mathbf{C}_n\right\}.$$

Then, up to a change of language (naming isomorphism types of members of $\mathbf{C}$), $\mathbf{C}^*$ is a super-robust cofinal sub-class of $\mathbf{C} = \bigcup_n \mathbf{C}_n$, and $T = T_{\mathbf{C}}$. □

4.3. **Some definable quotients.** We now turn to the question of whether a theory that is interpreted in the generic theory of a [semi-]tame class must also be the generic theory of a [semi-]tame class. For the sake of brevity, we restrict attention to certain "smooth" quotients that, by definition, respect the fact that the ambient theory arises from Fraïssé's construction. (If one considers two Lie geometries in a smoothly approximable theory, the non-orthogonality relation induces a smooth quotient.)

Definition 4.8. Let $\mathbf{C}$ be a [semi-]coherent class with generic model $\mathcal{M}$. Let $X \subseteq M^n$ be a 0-definable set, and let $E \subseteq X \times X$ be a 0-definable equivalence relation. We say that X/E *is a smooth quotient for* $\mathbf{C}$ if:

- For all $\bar{a} \in X$, $a_i \neq a_j$ whenever $i < j < n$.
- For every large enough $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$,
$$\{a_0, ..., a_{n-1} : \bar{a} \in X \cap \|\mathfrak{a}\|^n\} = \|\mathfrak{a}\|.$$
(Let $N_X < \omega$ be a threshold for "large enough" in this sense.)
- There is a monotone non-decreasing function $\xi_{X/E} : \omega \to \omega$ such that $\lim_{k\to\infty} \xi_{X/E}(k) = \infty$ and for all $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$,
$$\left|(X \cap \|\mathfrak{a}\|^n)/E\right| \geq \xi_{X/E}(|\mathfrak{b}|).$$

We define a language $\mathscr{L}_{X/E}$ as follows; in parallel, we describe how to convert $\mathcal{M}$ into an $\mathscr{L}_{X/E}$-structure, $\mathcal{M}_{X/E}$.

- $\|\mathcal{M}_{X/E}\| = X/E$.
- Suppose $\mathfrak{b} \in \mathbf{C}(\mathcal{M})$ and $q(\bar{x}_0, ..., \bar{x}_{k-1}, \bar{y}) = \mathrm{qftp}^{\mathcal{M}}(\bar{b}_0, ..., \bar{b}_{k-1}, \bar{c})$ where $\bar{b}_0, ..., \bar{b}_{k-1} \in X \cap \|\mathfrak{b}\|^n$, $\bar{c} \in \|\mathfrak{b}\|^{<\omega}$ satisfy the following:
  - $\bar{b}_i/E \neq \bar{b}_j/E$ whenever $i < j < k$;
  - $(X \cap \|\mathfrak{b}\|^n)/E = \{\bar{b}_0/E, ..., \bar{b}_{k-1}/E\}$;
  - $\bar{c}$ is an enumeration of $\|\mathfrak{b}\| \setminus \{b_{i,j}\}_{i<k, j<n}$.

  Then $R_{\mathfrak{b}^\bullet, q}$ is a k-ary relation symbol of $\mathscr{L}_{X/E}$ – where $\mathfrak{b}^\bullet$ denotes the $Aut(\mathcal{M})$-orbit of $\mathfrak{b}$ – and
$$\left(\bar{b}_0/E, ..., \bar{b}_{k-1}/E\right) \in R_{\mathfrak{b}^\bullet, q}^{\mathcal{M}_{X/E}}.$$

For $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$, we define $\mathfrak{a}_{X/E}$ to be the induced substructure of $\mathcal{M}$ with universe $(X \cap \|\mathfrak{a}\|^n)/E$. We define a class $\mathbf{C}_{X/E}$ of $\mathscr{L}_{X/E}$ to be the closure of

$$\left\{\mathfrak{a}_{X/E} : \mathfrak{a} \in \mathbf{C}(\mathcal{M}),\ |\mathfrak{a}| \geq N_X\right\}$$

under isomorphisms. Viewing $\mathbf{C}$ and $\mathbf{C}_{X/E}$ as categories under $\mathscr{L}$- and $\mathscr{L}_{X/E}$-embeddings, respectively, the mapping $\mathfrak{a} \mapsto \mathfrak{a}_{X/E}$ amounts to a functor $-_{X/E} : \mathbf{C} \to \mathbf{C}_{X/E}$.

To start with, we must observe that smooth quotients of a [semi-]coherent class yield [semi-]coherent classes. This is basically obvious from the definition of a smooth quotient, so we omit the proof.

Observation 4.9. Let $\mathbf{C}$ be a [semi-]coherent class with generic model $\mathcal{M}$, and let X/E be a smooth quotient for $\mathbf{C}$. Then the functor $-_{X/E} : \mathbf{C} \to \mathbf{C}_{X/E}$ is full. In particular, $\mathbf{C}_{X/E}$ is, again, a [semi-]coherent class with generic model $\mathcal{M}_{X/E}$.

Finally, verifying that smooth quotients of [semi-]tame classes are [semi-]tame classes amounts to little more than "fiddling" with certain formulas induced by the interpretation.

Proposition 4.10. *Let* $\mathbf{C}$ *be a [semi-]coherent class with generic model* $\mathcal{M}$*, and let* X/E *be a smooth quotient for* $\mathbf{C}$*. Then* $\mathbf{C}_{X/E}$ *is [semi-]tame.*

Proof. We begin by choosing a cofinal super-robust subclass $\mathbf{C}^*$ of $\mathbf{C}$; let $\nu : \mathscr{L} \to \omega$ be the associated grading of $\mathscr{L}$. Without loss of generality, we may assume that for all $\mathfrak{a}, \mathfrak{b} \in \mathbf{C}^*$, $|\mathfrak{a}| \geq N_X$, $\mathrm{rk}^{\mathbf{C}^*}(\mathfrak{a}) \geq \max\{\nu(X), \nu(E)\}$, and

$$\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) < \mathrm{rk}^{\mathbf{C}}(\mathfrak{b}) \;\Rightarrow\; \left|(X \cap \|\mathfrak{a}\|^n)/E\right| < \left|(X \cap \|\mathfrak{b}\|^n)/E\right|.$$

We then define $\mathbf{C}^*_{X/E} = \left\{\mathfrak{a}_{X/E} : \mathfrak{a} \in \mathbf{C}^*\right\}$. To see that $\mathbf{C}^*_{X/E}$ is super-robust, it is essentially enough to observe that there is a transformation

$$\mathscr{L}_{X/E} \to \mathscr{L} : \varphi(v_0, ..., v_{k-1}) \mapsto \check{\varphi}(\overline{x}_0, ..., \overline{x}_{k-1})$$

such that for all $\mathfrak{b} \in \mathbf{C}^*(\mathcal{M})$ and $\overline{b}_0, ..., \overline{b}_{k-1} \in X \cap \|\mathfrak{b}\|^n$, if $\mathrm{rk}^{\mathbf{C}^*}(\mathfrak{b}) \geq \nu(\check{\varphi})$, then,

$$\begin{tikzcd}
\mathcal{M}_{X/E} \models \varphi\left(\overline{b}_0/E, ..., \overline{b}_{k-1}/E\right) \arrow[r, Leftrightarrow] \arrow[d, dotted, leftrightarrow] & \mathcal{M} \models \check{\varphi}(\overline{b}_0, ..., \overline{b}_{k-1}) \arrow[d, Leftrightarrow] \\
\mathfrak{b}_{X/E} \models \varphi\left(\overline{b}_0/E, ..., \overline{b}_{k-1}/E\right) \arrow[r, Leftrightarrow] & \mathfrak{b} \models \check{\varphi}(\overline{b}_0, ..., \overline{b}_{k-1}).
\end{tikzcd}$$

□

4.4. **Sums and products.** In this penultimate subsection, we consider two constructions that are important in discussions of "algorithmic meta-theorems" as understood in finite model theory and computational complexity theory more generally. Namely, we consider direct products of structures and direct sums (i.e. disjoint unions) of structures. In both cases, we complicate the situation somewhat by allowing "fusion" over a common substructure rather than appealing to Observation 4.1. In finite model theory, it seems not to be the standard to blithely add constants to reduce all questions of definability to 0-definability, and the decision to consider fused products and sums explicitly just keeps our development here in line with that trend.

Obviously, we should start our discussion by pinning down the definitions of fused products and sums of structures.

Definition 4.11 (Fused sums and products)**.** Let $\mathscr{L}$ be a first-order language with no function symbols, and let $\mathcal{A}_0, ..., \mathcal{A}_{N-1}, \mathcal{B}$ be a family of $\mathscr{L}$-structures such that for each $t < N$, $\mathcal{B} \leq \mathcal{A}_t$, and for all $s < t < N$, $A_s \cap A_t = B$.

- We write $\prod_{t<N} A_t/B$ as shorthand for the set,

$$\left(\left(\prod\nolimits_{t<N} A_t\right) \setminus B^N\right) \cup \{\underbrace{(b, b, ..., b)}_{N} : b \in B\}.$$

  Then the structure $\mathcal{N} = \prod_{t<N} \mathcal{A}_t/\mathcal{B}$ has universe $\prod_{t<N} A_t/B$ and interpretations,

$$c^{\mathcal{N}} = (c^{\mathcal{A}_0}, ..., c^{\mathcal{A}_{N-1}}) = (\underbrace{c^{\mathcal{B}}, c^{\mathcal{B}}, ..., c^{\mathcal{B}}}_{N})$$

  for each constant symbol c, and

$$R^{\mathcal{N}} = \left\{(\overline{a}_0, ..., \overline{a}_{k-1}) : \bigwedge_{t<N} (a_{0,t}, ..., a_{k-1,t}) \in R^{\mathcal{A}_t}\right\}$$

for each k-ary relation symbol of $\mathscr{L}$.

- The structure $\mathcal{N} = \coprod_{t<N} \mathcal{A}_t/\mathcal{B}$ has universe $\bigcup_{t<N} A_t$ and interpretations, $c^{\mathcal{N}} = c^{\mathcal{B}}$ for each constant symbol c, and
$$R^{\mathcal{N}} = \bigcup_{t<N} R^{\mathcal{A}_t}$$
for each relation symbol R of $\mathscr{L}$.

In the proof of the Theorem 4.13 below, the key "trick" is just appealing to an old theorem of Feferman and Vaught, in two forms. Good sources of proofs and discussion of this theorem are [17] and [26].

Theorem (Feferman-Vaught for fused products)**.** *Let $\mathscr{L}$ be a first-order language with no function symbols, and let $0 < N < \omega$. For every $\mathscr{L}$-formula $\varphi(x_0, ..., x_{n-1})$, there are, for each $S \subseteq n$, a quantifier-free formula $\sigma^{\times}_{\varphi,S}(z_0, ..., z_{k-1})$ in the language of boolean algebras and $\mathscr{L}$-formulas $\psi^0_{\varphi,S}(\overline{x}), ..., \psi^{k-1}_{\varphi,S}(\overline{x})$ such that for any family $\mathcal{A}_0, ..., \mathcal{A}_{N-1}, \mathcal{B}$ of $\mathscr{L}$-structures, if $\mathcal{B} \leq \mathcal{A}_t$ and $A_s \cap A_t = B$ whenever $s, t < N$, $s \neq t$, then the following holds:*

Given $\overline{a} \in \left(\prod_{t<N} A_t/B\right)^n$, let $S = \{j < n : a_{j,0} = \cdots = a_{j,N-1} = b \in B\}$, and set
$$\left[\psi^i_{\varphi,S}, (\mathcal{A}_t)_t, \mathcal{B}, \overline{a}\right] := \left\{t < N : \mathcal{A}_t \vDash \psi^i_{\varphi,S}(a_{0,t}, a_{1,t}, ..., a_{n-1,t})\right\}$$
for each $i < k$. Then,
$$\prod\nolimits_{t<N} \mathcal{A}_t/\mathcal{B} \vDash \varphi(\overline{a}) \;\Leftrightarrow\; \mathcal{P}(N) \vDash \sigma^{\times}_{\varphi,S}\left([\psi^0_{\varphi,S}, (\mathcal{A}_t)_t, \mathcal{B}, \overline{a}], ..., [\psi^{k-1}_{\varphi,S}, (\mathcal{A}_t)_t, \mathcal{B}, \overline{a}]\right).$$

Theorem (Feferman-Vaught for fused sums)**.** *Let $\mathscr{L}$ be a first-order language with no function symbols, and let $0 < N < \omega$. For every $\mathscr{L}$-formula $\varphi(x_0, ..., x_{n-1})$ and each partial function $f : n \rightharpoonup N$, there $\mathscr{L}$-formulas*
$$\psi^{t,0}_{\varphi,f}(\overline{x}_{\restriction f^{-1}(t)}, \overline{x}_{\restriction n\setminus \mathrm{dom}(f)}), ..., \psi^{t,k_t-1}_{\varphi,f}(\overline{x}_{\restriction f^{-1}(t)}, \overline{x}_{\restriction n\setminus \mathrm{dom}(f)})$$
for each $t < N$ and a quantifier-free formula, $\sigma^{\sqcup}_{\varphi,f}\left(z_{0,0}, ..., z_{0,k_0-1}, ..., z_{N-1,0}, ..., z_{N-1,k_{N-1}-1}\right)$ such that for any family $\mathcal{A}_0, ..., \mathcal{A}_{N-1}, \mathcal{B}$ of $\mathscr{L}$-structures, if $\mathcal{B} \leq \mathcal{A}_t$ and $A_s \cap A_t = B$ whenever $s, t < N$, $s \neq t$, then the following holds:

Given $\overline{a} \in \left(\bigcup_{t<N} A_t\right)^n$, let $f : n \rightharpoonup N$ by the unique partial function such that $f(i) = t \Leftrightarrow a_i \in A_t \setminus B$, and for each $t < N$ and $i < k_t$, set
$$\left[\psi^{t,i}_{\varphi,f}, (\mathcal{A}_t)_t, \mathcal{B}, \overline{a}\right] := \begin{cases} \{\emptyset\} & \text{if } \mathcal{A}_t \vDash \psi^{t,i}_{\varphi,f}(\overline{a}_{\restriction f^{-1}(t)}, \overline{a}_{\restriction n\setminus \mathrm{dom}(f)}) \\ \emptyset & \text{if } \mathcal{A}_t \vDash \neg\psi^{t,i}_{\varphi,f}(\overline{a}_{\restriction f^{-1}(t)}, \overline{a}_{\restriction n\setminus \mathrm{dom}(f)}). \end{cases}$$
Then,
$$\coprod\nolimits_{t<N} \mathcal{A}_t/\mathcal{B} \vDash \varphi(\overline{a}) \;\Leftrightarrow\; \mathcal{P}(2) \vDash \sigma^{\sqcup}_{\varphi,f}\left(\left([\psi^{t,i}_{\varphi,f}, (\mathcal{A}_t)_t, \mathcal{B}, \overline{a}]\right)_{t<N, i<k_t}\right).$$

Now, we return to the context of classes of finite structures, defining what is meant by a fused product or fused sum of several classes over a fixed finite structure $\mathfrak{e}$.

Definition 4.12. Fix a single finite $\mathscr{L}$-structure $\mathfrak{e}$. Let $0 < N < \omega$, and for each $t < N$, let $\mathbf{C}_t$ be either a [semi-]coherent class of $\mathscr{L}$-structures or a singleton. Further, assume that $\mathfrak{e} \leq^* \mathfrak{a}$ for each $\mathfrak{a} \in \mathbf{C}_t$ and that at least one of $\mathbf{C}_0, ..., \mathbf{C}_{N-1}$ is a proper class.. We define the fused sum $\coprod_{t<N} \mathbf{C}_t/\mathfrak{e}$ and the fused product $\prod_{t<N} \mathbf{C}_t/\mathfrak{e}$ as follows:

- Up to isomorphism $\coprod_{t<N} \mathbf{C}_t/\mathfrak{e}$ is the class of finite $\mathscr{L}$-structures of the form $\coprod_{t<N} \mathfrak{a}_t/\mathfrak{e}$, where $\mathfrak{a}_t \in \mathbf{C}_t$ and $\mathfrak{e} \leq \mathfrak{a}_t$ for each $t < N$, and $\|\mathfrak{a}_s\| \cap \|\mathfrak{a}_t\| = \|\mathfrak{e}\|$ whenever $s < t < N$.
- Up to isomorphism $\prod_{t<N} \mathbf{C}_t/\mathfrak{e}$ is the class of finite $\mathscr{L}$-structures of the form $\prod_{t<N} \mathfrak{a}_t/\mathfrak{e}$, where $\mathfrak{a}_t \in \mathbf{C}_t$ and $\mathfrak{e} \leq \mathfrak{a}_t$ for each $t < N$, and $\|\mathfrak{a}_s\| \cap \|\mathfrak{a}_t\| = \|\mathfrak{e}\|$ whenever $s < t < N$.

Finally, we state and prove the theorem on finite products and sum. (Unfortunately, this theorem alone is not enough to give us a result of the form "if **C** is coordinatized by [semi-]tame classes, then **C** is [semi-]tame"; for that, we would need to address (i) some kind of uniform interaction between the components of a sum, and (ii) something like an infinite direct sum. While that kind of result does not seem implausible, it would be, we believe, a long development in itself.)

Theorem 4.13. *Consider a single finite $\mathscr{L}$-structure $\mathfrak{e}$. Let $0 < N < \omega$, and for each $t < N$, let $\mathbf{C}_t$ be either a [semi-]tame class of $\mathscr{L}$-structures or a singleton. Further, assume that for each $\mathfrak{a} \in \mathbf{C}_t$ $(t < N)$, there is a* unique *embedding $\mathfrak{e} \to \mathfrak{a}$ up to automorphisms of $\mathfrak{e}$, and that at least one of $\mathbf{C}_0, ..., \mathbf{C}_{N-1}$ is a proper class. Then, $\prod_{t<N} \mathbf{C}_t/\mathfrak{e}$ and $\coprod_{t<N} \mathbf{C}_t/\mathfrak{e}$ are both [semi-]tame.*

Proof for $\prod$. For brevity, we write $\mathbf{D}$ in place of $\prod_{t<N} \mathbf{C}_t/\mathfrak{e}$. We will also, for economy, assume that $\mathbf{C}_0, ..., \mathbf{C}_{N-1}$ are *themselves* super-robust via $\nu_0, ..., \nu_{N-1} : \mathscr{L} \to \omega$, respectively. (Only very minor alterations are required when some of $\mathbf{C}_t$'s may be singletons.) We may certainly choose a cofinal subclass $\mathbf{D}^*$ of $\mathbf{D}$ for which there is a strictly increasing function $f : \omega \to \omega$ such that

$$\mathrm{rk}^{\mathbf{D}^*}\left(\prod\nolimits_{t<N} \mathfrak{a}_t/\mathfrak{b}\right) \geq n \;\Rightarrow\; \bigwedge\nolimits_{t<N} \mathrm{rk}^{\mathbf{C}_t}(\mathfrak{a}_t) \geq f(n).$$

We then define a grading $\nu : \mathscr{L} \to \omega$ for $\mathbf{D}^*$ by setting

$$\nu(\varphi) = \min\left\{n : \bigwedge\nolimits_{t<N}\bigwedge\nolimits_{i<k}\bigwedge\nolimits_{S}\nu_t(\psi^i_{\varphi,S}) \geq f(n)\right\}.$$

where $\sigma^\times_{\varphi,S}(z_0, ..., z_{k-1})$ and $\psi^0_{\varphi,S}(\overline{x})$, ..., $\psi^{k-1}_{\varphi,S}(\overline{x})$ are as granted by the Feferman-Vaught Theorem for fused products. Of course, we must demonstrate that ν is actually a witness to the super-robustness of $\mathbf{D}^*$.

So, let $\prod_{t<N} \mathfrak{a}_t/\mathfrak{e} \leq \prod_{t<N} \mathfrak{a}'_t/\mathfrak{e}'$ be members of $\mathbf{D}^*$, where $\mathfrak{e} \cong \mathfrak{e}'$, $\varphi(x_0, ..., x_{n-1}) \in \mathscr{L}$, and $\overline{a} \in (\prod_t \|\mathfrak{a}_t\|/\|\mathfrak{e}\|)^n$, and suppose $\mathrm{rk}\left(\prod_{t<N} \mathfrak{a}_t/\mathfrak{e}\right) \geq \nu(\varphi)$. Let

$$S = \{j < n : a_{j,0} = \cdots = a_{j,N-1} = b \in \|\mathfrak{e}\|\},$$

$$S' = \left\{j < n : a_{j,0} = \cdots = a_{j,N-1} = b \in \|\mathfrak{e}'\|\right\}.$$

By the uniqueness assumption on embeddings of $\mathfrak{e}$ into members of $\mathbf{C}_t$ $(t < N)$, we have $S = S'$. Next, let $\sigma^\times_{\varphi,S}(z_0, ..., z_{k-1})$ and $\psi^0_{\varphi,S}(\overline{x})$, ..., $\psi^{k-1}_{\varphi,S}(\overline{x})$ be as granted by the Feferman-Vaught Theorem for fused products. For brevity, let

$$X_i = \left[\psi^i_{\varphi,S}, (\mathfrak{a}_t)_t, \mathfrak{e}, \overline{a}\right] := \left\{t < N : \mathfrak{a}_t \vDash \psi^i_{\varphi,S}(a_{0,t}, a_{1,t}, ..., a_{n-1,t})\right\}$$

$$X'_i = \left[\psi^i_{\varphi,S}, (\mathfrak{a}'_t)_t, \mathfrak{e}', \overline{a}\right] := \left\{t < N : \mathfrak{a}'_t \vDash \psi^i_{\varphi,S}(a_{0,t}, a_{1,t}, ..., a_{n-1,t})\right\}$$

for each $i < k$. Since $\mathrm{rk}^{\mathbf{C}_t}(\mathfrak{a}_t) \geq \nu_t(\psi^i_{\varphi,S})$ for each $t < N$ and $i < k$, we find that $X_i = X'_i$ for each $i < k$. Thus,

$$\begin{aligned}
\prod\nolimits_{t<N} \mathfrak{a}_t/\mathfrak{e} \vDash \varphi(\overline{a}) \;&\Leftrightarrow\; \mathcal{P}(N) \vDash \sigma^\times_{\varphi,S}(X_0, ..., X_{k-1}) \\
&\Leftrightarrow\; \mathcal{P}(N) \vDash \sigma^\times_{\varphi,S}(X'_0, ..., X'_{k-1}) \\
&\Leftrightarrow\; \prod\nolimits_{t<N} \mathfrak{a}'_t/\mathfrak{e}' \vDash \varphi(\overline{a})
\end{aligned}$$

as desired. (The proof of the theorem for fused sums is not interestingly different from the argument for fused products, so in the interests of brevity, we omit it.) □

5. Stability-theoretic properties of tame classes

In this section, we concentrate more specifically on coherent and tame classes, and in this context, we examine some generalized-stability-theoretic (or "geometric") properties of classes of finite structures. Firstly, we examine one branch of the criterion for instabilty – that any unstable theory has either the Independence Property or the Strict-order Propery. We will see that the generic theory of a tame class can *never* have the Strict-order Property, so any tame class whose generic theory is NIP must be stable. Secondly, we give a topological, or Baire categorical, characterization of þ-independence (for triples of finite sets) in terms of what we call "locally pro-generic" types. Using technology from [14] (so requiring an additional hypothesis of "partial elimination of imaginaries"), we find that for the generic theory of a tame class, rosiness implies super-rosiness with finite $U^{\text{þ}}$-rank.

5.1. NSOP.

We begin this subsection by restating the definition of the Strict-order Property, and after that (and without further preamble), we state and prove the only result – that tame classes are necessarily NSOP.

Definition 5.1. Let T be an $\mathscr{L}$-theory. We say that T has the Strict-order Property if there are a formula $\varphi(x_0, ..., x_{m-1}, y_0, ..., y_{n-1})$, a model $\mathcal{M} \vDash T$, and a sequence $\bar{b}_0, \bar{b}_1, ..., \bar{b}_i, ...$ in M^n such that $\varphi(\mathcal{M}, \bar{b}_i) \subsetneq \varphi(\mathcal{M}, \bar{b}_j)$ whenever $i < j < \omega$.

Proposition 5.2. *Let* $\mathbf{C}$ *be a coherent class. If* $\mathbf{C}$ *is tame, then* $T_{\mathbf{C}}$ *does not have the Strict-order Property.*

Proof. Suppose $\mathbf{C}$ is tame with a cofinal super-robust subclass $\mathbf{C}^*$. Towards a contradiction, suppose $\varphi(x_0, ..., x_{m-1}, y_0, ..., y_{n-1})$ witnesses the fact that $T_{\mathbf{C}}$ has the Strict-order Property. Using $\aleph_0$-categoricity and taking $\mathcal{M}$ to be the generic model of $\mathbf{C}$, we may assume that there are $\bar{b}_0, \bar{b}_1, ..., \bar{b}_i, ...$ in M^n such that $\varphi(\mathcal{M}, \bar{b}_i) \subsetneq \varphi(\mathcal{M}, \bar{b}_j)$ whenever $i < j < \omega$. Let $E_\varphi(\bar{y}_1, \bar{y}_2)$ and $\bar{y}_1 \leq_\varphi \bar{y}_2$ be the formulas,

$$\forall \bar{x} \left(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2)\right) \text{ and } \forall \bar{x} \left(\varphi(\bar{x}, \bar{y}_1) \rightarrow \varphi(\bar{x}, \bar{y}_2)\right),$$

respectively. Now, again by $\aleph_0$-categoricity,

$$T_{\mathbf{C}} \vDash \exists \bar{z}_1, \bar{z}_2 \forall \bar{y}_1, \bar{y}_2 \left(\bar{z}_1 <_\varphi \bar{y}_1 <_\varphi \bar{y}_2 <_\varphi \bar{z}_2 \rightarrow \exists \bar{y} \left(\bar{y}_1 <_\varphi \bar{y} <_\varphi \bar{y}_2\right)\right)$$

– that is, the partial order $(M^n/E_\varphi, \leq_\varphi)$ has a dense interval. Then, for $\mathfrak{a} \in \mathbf{C}^*$ of large enough rank, $(\|\mathfrak{a}\|^n/E_\varphi, \leq_\varphi)$ must also have a dense interval – which is nonsense for a finite partial order. $\square$

Since an unstable theory must have either SOP or IP, we immediately derive the following corollary.

Corollary 5.3. *Let* $\mathbf{C}$ *be a tame class. If* $T_{\mathbf{C}}$ *is NIP, then* $T_{\mathbf{C}}$ *is stable.*

Remark 5.4. There is a second, slightly different, approach to proving that tame classes engender generic theories that are NSOP, and we present this approach in Appendix B. This approach goes through the notion of "anti-proficiency" defined in [27]. Alone this alternative does not seem especially interesting, but it does illuminate a general relationship between tameness and proofs of 0,1-laws for inflationary fixed-point logic (IFP) along the lines of that first given by [5].

5.2. Independence.

In this subsection, we will develop a notion of "locally pro-generic" type that allows us to construct a model-theoretic notion of independence using Baire category theory as mentioned in Subsection 2.4 and as developed more fully in [20]. The notion of independence we develop, denoted $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{g}}$, is applicable for triples of finite subsets in the generic model of any coherent class and provides an alternative formulation of þ-independence. In consequence, we will also find, using a result from [14], that when the class is tame and $T_{\mathbf{C}}$ eliminates imaginaries, the symmetry of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{g}}$ characterizes super-rosiness of the generic theory.

Throughout this subsection, $\mathbf{C}$ is a coherent class of $\mathscr{L}$-structures with generic model $\mathcal{M}$. We will further assume that for every $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$, $\mathrm{acl}^{\mathcal{M}}(\|\mathfrak{c}\|) = \|\mathfrak{c}\|$.

The first step in the project of defining and characterizing $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{g}}$ is to set up a family of languages $\mathscr{L}^k_{\overline{a}/\mathfrak{c}}$ expanding $\mathscr{L}$ and corresponding Baire spaces $\mathbb{X}^k(\overline{a}/\mathfrak{c})$ of expansions of $\mathcal{M}$. The members of $\mathbb{X}^k(\overline{a}/\mathfrak{c})$ will correspond to certain complete types over $\mathcal{M}$.

Definition 5.5. Let $0 < n < \omega$, $\overline{a} \in M^n$, and $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$. For $k < \omega$, we define $\mathscr{L}^k_{\overline{a}/\mathfrak{c}}$ to be the language with signature,

$$\mathrm{sig}(\mathscr{L}^k_{\overline{a}/\mathfrak{c}}) = \mathrm{sig}(\mathscr{L}) \cup \|\mathfrak{c}\| \cup \left\{ R^{(\ell)}_{\varphi} : \begin{array}{l} \ell \leq k,\ m \leq n \\ \varphi(x_{i_0}, ..., x_{i_{m-1}}, y_0, ..., y_{\ell-1}) \in \mathscr{L}(\|\mathfrak{c}\|), \\ \nu(\varphi) \leq k, \\ \mathcal{M} \models \exists \overline{y}\, \varphi(\overline{a}, \overline{y}) \end{array} \right\}$$

where $\|\mathfrak{c}\|$ is understood as a set of new constant symbols. Writing $S(\overline{a}/\mathfrak{c})$ for the set of complete types $p \in S_n(\mathcal{M})$ extending $\mathrm{tp}(\overline{a}/\|\mathfrak{c}\|)$, each $p \in S(\overline{a}/\mathfrak{c})$ determines an $\mathscr{L}^k_{\overline{a}/\mathfrak{c}}$-expansion $\mathcal{M}^k_p$ of $\mathcal{M}$ – setting $c^{\mathcal{M}^k_p} = c$ for each $c \in \|\mathfrak{c}\|$ and

$$R^{\mathcal{M}^k_p}_{\varphi} = \left\{ \overline{b} \in M^{\ell} : \varphi(\overline{x}, \overline{b}) \in p \right\}$$

whenever $\ell \leq k$, $\varphi(x_0, ..., x_{n-1}, y_0, ..., y_{\ell-1}) \in \mathscr{L}(\|\mathfrak{c}\|)$, and $\nu(\varphi) \leq k$. We define $\mathbb{X}^k(\overline{a}/\mathfrak{c})$ to be the set of all such expansions of $\mathcal{M}$, observing that this is a closed subset of the topological space of all $\mathscr{L}^k_{\overline{a}/\mathfrak{c}}$-expansions of $\mathcal{M}$.

It isn't hard to see that $Aut(\mathcal{M}/\mathfrak{c})$ acts on $\mathbb{X}^k(\overline{a}/\mathfrak{c})$ by homeomorphisms and further, that taking reducts induces a continuous surjective function $\mathbb{X}^{k_2}(\overline{a}/\mathfrak{c}) \to \mathbb{X}^{k_1}(\overline{a}/\mathfrak{c})$ whenever $k_1 \leq k_2 < \omega$. From the latter observation, we find ourselves with an inverse system, and we can construct its inverse limit $\mathbb{X}^{\infty}(\overline{a}/\mathfrak{c})$ with continuous projection maps $\pi_k : \mathbb{X}^{\infty}(\overline{a}/\mathfrak{c}) \to \mathbb{X}^k(\overline{a}/\mathfrak{c})$ for each $k < \omega$. It is also not hard to see that there is a bijective correspondence between $\mathbb{X}^{\infty}(\overline{a}/\mathfrak{c})$ and the closed set $S(\overline{a}/\mathfrak{c})$ of types over $\mathcal{M}$ extending $\mathrm{tp}(\overline{a}/\mathfrak{c})$: $p \mapsto \left(\mathcal{M}^k_p\right)_{k<\omega}/\!\sim$, where $\sim$ is the equivalence relation involved in the construction of the inverse limit from the direct product $\prod_k \mathbb{X}^k(\overline{a}/\mathfrak{c})$.

In the following commutative diagram, we consider parameters $k_1 \leq k_2 < \omega$, $\overline{a}, \overline{b} \in M^{<\omega}$, $\mathfrak{c}_0, \mathfrak{c}_1 \in \mathbf{C}(\mathcal{M})$ such that $\mathfrak{c}_0 \leq \mathfrak{c}_1$. The commutativity of the following diagram captures, for the most part, the pith of Definition 5.5.

$$\begin{tikzcd}
\mathbb{X}^{k_2}(\overline{a}\overline{b}/\mathfrak{c}_1) \arrow[r] \arrow[d] & \mathbb{X}^{k_2}(\overline{a}/\mathfrak{c}_1) \arrow[r] \arrow[d] & \mathbb{X}^{k_2}(\overline{a}/\mathfrak{c}_0) \arrow[d] \\
\mathbb{X}^{k_1}(\overline{a}\overline{b}/\mathfrak{c}_1) \arrow[r] & \mathbb{X}^{k_1}(\overline{a}/\mathfrak{c}_1) \arrow[r] & \mathbb{X}^{k_1}(\overline{a}/\mathfrak{c}_0)
\end{tikzcd}$$

Definition 5.6. Let $0 < n < \omega$, $\overline{a} \in M^n$, and $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$. We say that $p \in S(\overline{a}/\mathfrak{c})$ *is a locally pro-generic extension of* $\mathrm{tp}(\overline{a}/\mathfrak{c})$ if

$$\left\{ k < \omega : \begin{array}{l} \text{the } Aut(\mathcal{M}/\mathfrak{c})\text{-orbit of } \mathcal{M}^k_p \text{ is co-meager} \\ \text{in a non-empty } Aut(\mathcal{M}/\mathfrak{c})\text{-invariant} \\ G_{\delta} \text{ subset of } \mathbb{X}^k(\overline{a}/\mathfrak{c}) \end{array} \right\}$$

is infinite. For $B \subset_{\mathrm{fin}} M$, we write $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{g}}_{\mathfrak{c}} B$ to mean that there is a locally pro-generic extension $p \in S(\overline{a}/\mathfrak{c})$ such that $\mathrm{tp}(\overline{a}/B\mathfrak{c}) \subset p$.

To replace $\mathfrak{c}$ with an arbitrary base set, we proceed as follows: for $B, C \subset_{\text{fin}} M$ and $\overline{a} \in M^n$ $(0 < n < \omega)$, we assert $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$ if for all $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$, if $C \subseteq \|\mathfrak{c}\|$, then there is some $B' \equiv_{\overline{a}C} B$ such that $\overline{a} \cap \|\mathfrak{c}\| = \overline{a} \cap \operatorname{acl}(C)$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{\mathfrak{c}} B'$. Finally, and as usual, we write $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$ to mean that for some enumeration $\overline{a}$ of A, $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$.

From the definition of $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}$ and the commutative diagram that preceded it, the following list of properties of $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}$ are, more or less, immediate.

Fact 5.7. $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}$ has the following properties:

Invariance: Let $A, B, C \subset_{\text{fin}} M$, and let $g \in Aut(\mathcal{M})$. Then $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$ if and only if $gA \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{gC} gB$.

Preservation of algebraic dependence: Suppose $A, B, C \subset_{\text{fin}} M$; then, if $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$, then $\operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl}(C)$.

Partial right-transitivity: Suppose $A \subset_{\text{fin}} \mathbb{M}$ and $B, C, D \subset \mathbb{M}$; then, if $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{D} BC$, then $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{D} C$ and $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{DC} B$.

Partial left-transitivity: Suppose $A_1, A_2 \subset_{\text{fin}} \mathbb{M}$ and $B, C \subset \mathbb{M}$; then, if $A_1 \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$ and $A_2 \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{CA_1} B$, then $A_1 A_2 \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$.

Lemma 5.8. *Extension properties of* $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}$

1. *(Extension) Let $\overline{a} \in M^{<\omega}$ and $C \subseteq B \subseteq B' \subset_{fin} M$; if $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$, then there is an $\overline{a}' \in M^{<\omega}$ such that $\overline{a}' \equiv_B \overline{a}$ and $\overline{a}' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B'$.*
2. *(Base extension) Let $\overline{a} \in M^{<\omega}$ and $C \subseteq C' \subseteq B \subset_{fin} M$; if $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$, then there are $B' \subset_{fin} M$ and $\overline{a}' \in M^{<\omega}$ such that $\overline{a}'B' \equiv_C \overline{a}B$ and $\overline{a}' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C'} B'$.*

Proof. For 5.8:1, one just observes that the locally pro-generic extension witnessing $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B$ for some $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$ containing C also witnesses $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}_{C} B'$ for that $\mathfrak{c}$. For 5.8:2, one just observes that any $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$ that contains C' also contains C and then applies Invariance. $\square$

The following lemma is actually an immediate consequence of the "þ-Independence is weakest" theorem proved in [9] and the facts we've already noted about $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}$. However, it also seems worth noting that it can be proven directly from the definition of $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}$, and in any case, its purpose is to deal with one direction of the equivalence stated in Proposition 5.10, proved immediately after.

Lemma 5.9. *Let $\overline{a} \in M^{<\omega}$ and $B, C \subset_{fin} M$, and suppose $\operatorname{tp}(\overline{a}/BC)$ does not þ-fork over C. There is an extension $p \in S_n(\mathcal{M})$ of $\operatorname{tp}(\overline{a}/BC)$ such that for every $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$ containing C and every large enough $k < \omega$, if $\overline{a} \cap \operatorname{acl}(C) = \overline{a} \cap \|\mathfrak{c}\|$, then*

$$\mathbf{C}^k(p, \mathfrak{c}) = \left\{ g\mathfrak{b} : \mathfrak{b} <_{fin} \mathcal{M}_p^k,\ g \in Aut(\mathcal{M}/\mathfrak{c}) \right\}$$

has JEP and AP.

Sketch. One simply observes that any obstruction to JEP or AP in some $\mathbf{C}^k(p, \mathfrak{c})$ would amount to an implication $\operatorname{tp}(\overline{a}/BC) \models \bigvee_{i<N} \varphi_i(\overline{x})$, where each $\varphi_i \in \mathscr{L}^{\text{eq}}(M)$ $(i < N)$ þ-divides over C. $\square$

Proposition 5.10. $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{g}}$ *is identical to þ-independence for triples of finite sets of real elements.*

Proof. Obviously, there are two claims to prove, which we deal with in tandem.

Claim. Let $\overline{a} \in M^{<\omega}$ and $B, C \subset_{\text{fin}} M$. If $\text{tp}(\overline{a}/BC)$ does not þ-fork over C, then $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{g}}_{C} B$.

Proof of claim. Using the previous lemma and the characterization in [20] of the existence of a co-meager orbit. (Again, one could just cite [9].) $\square$

Claim. Let $\overline{a} \in M^{<\omega}$ and $B, C \subset_{\text{fin}} M$. If $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{g}}_{C} B$, then $\text{tp}(\overline{a}/BC)$ does not þ-fork over C.

Proof. For the contrapositive, suppose $\text{tp}(\overline{a}/BC)$ þ-forks over C. Taking $\overline{b}$ as an enumeration of B, following manipulations in [9] and using $\aleph_0$-categoricity, we may choose some e in $\mathcal{M}^{\text{eq}}$ and a formula $\varphi(\overline{x}, \overline{y}, z) \in \mathscr{L}^{\text{eq}}(C)$ that isolates $\text{tp}(\overline{a}\overline{b}e/C)$ such that $\varphi(\overline{x}, \overline{b}, e)$ strong-divides in $\text{tp}(\overline{b}/Ce)$. We may also assume that $\overline{b} \cap \text{acl}(Ce) = \emptyset$. By definition $e = \overline{e}/E$ for some 0-definable equivalence relation and some $\overline{e} \in M^{<\omega}$. We choose $\mathfrak{c} \in \mathbf{C}(\mathcal{M})$ such that $C\overline{e} \subseteq \|\mathfrak{c}\|$ and $\mathfrak{b} \in \mathbf{C}(\mathcal{M})$ such that $\|\mathfrak{c}\| \cup \overline{b} \subseteq \|\mathfrak{b}\|$ and $\overline{b} \cap \|\mathfrak{c}\| = \emptyset$. As $\overline{b}$ avoids the algebraic closure of $C\overline{e}$ but is contained in the algebraic closure of $C\overline{e}\overline{a}$, we encounter a contradiction to AP in $\mathbf{C}^k(p, \mathfrak{c})$ for all k and $p \in S(\mathcal{M})$ extending $\text{tp}(\overline{a}/C\overline{b}e)$ – showing that $\overline{a} \not\mathop{\smile\!\!\!\!\!\!\mid}^{\text{g}}_{\mathfrak{c}} B$. $\square$

$\square$

When $T_{\mathbf{C}}$ happens to eliminates imaginaries, Proposition 5.10 amounts to a slightly novel characterization of rosiness.

Theorem 5.11. *Suppose $T_{\mathbf{C}}$ eliminates imaginaries. The following are equivalent:*

(1) $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{g}}$ is symmetric.
(2) $T_{\mathbf{C}}$ is rosy.
(3) $T_{\mathbf{C}}$ is super-rosy of finite $U^{þ}$-rank.

Proof. 1⇔2 is from [28] together with Proposition 5.10, and 2⇔3 is a result of [14]. $\square$

Remark 5.12. Let T be a theory in finitely-many sorts (which we call its *real* sorts), and let $\mathcal{N} \models T$. Then, T is said to have *partial elimination of imaginaries* if there is a finite set of 0-definable equivalence relations $E_0, ... E_{n-1}$ such that the reduct of $\mathcal{N}^{\text{eq}}$ to the real sorts and $\{\mathsf{S}_{E_0}, ..., \mathsf{S}_{E_{n-1}}\}$ uniformly eliminates imaginaries.

Partial elimination of imaginaries was an important hypothesis of [14] essentially because one can simply add finitely many imaginary sorts of $T_{\mathbf{C}}$ into the language of $\mathbf{C}$ while retaining tameness. Thus, the previous theorem applies to tame classes $\mathbf{C}$ that partially eliminate imaginaries (meaning that $T_{\mathbf{C}}$ does so) with little meaningful change.

6. Semi-tameness and 0,1-laws for first-order logic

In Section 3, we defined the property of generic-categoricity up to foundation rank.[4] At the time, we remarked in passing that generic-categoricity is, naïvely anyway, the closest one might expect to come to formulating a 0,1-law without making any reference to asymptotic probabilities. In this section, we will demonstrate that this similarity is not entirely accidental; indeed the 0,1-law for first-order logic is a sufficient condition for semi-tameness, as expressed in the following theorem.

Theorem 6.1. *Let $\mathbf{C}$ be a Fraïssé class such that algebraic closure is trivial in models of $T_{\mathbf{C}}$. If $\mathbf{C}$ has the (labeled) 0,1-law for first-order logic under uniform distributions and $T^{\text{as}}_{\mathbf{C}} = T_{\mathbf{C}}$, where $T^{\text{as}}_{\mathbf{C}}$ denotes the asymptotically almost-sure theory of $\mathbf{C}$, then $\mathbf{C}$ is tame.*

[4]Namely, we said that a semi-coherent class $\mathbf{C}$ is generically-categorical up to foundation rank if for any set of representatives $\{\mathfrak{a}_k\}_k$ of the isomorphism types in $\mathbf{C}$ and any non-principal ultrafilter $\mathscr{U}$ on ω, if $\{k : \text{rk}^{\mathbf{C}}(\mathfrak{a}) \geq e\} \in \mathscr{U}$ for every $e < \omega$, then $\prod_k \mathfrak{a}_k/\mathscr{U} \models T_{\mathbf{C}}$.

An important step in the proof of Theorem 6.1 is a move to "softer" formulation of the 0,1-law in which probabilities are evaluated *within* well-chosen members of $\mathbf{C}(\mathcal{M})$, where $\mathcal{M}$ is the generic model of $\mathbf{C}$. That is, we recover Theorem 6.1 as an immediate corollary of Lemma 6.4 and Proposition 6.6 below.

This relaxed formulation is much more obviously compatible with ultrafilters extending $\mathsf{Cone}_{\mathbf{C}}(\mathcal{M})$ or $\mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathcal{M})$, and the proof after making this shift is a fairly natural "paring down" process using an appropriate filter whose existence is guaranteed by the 0,1-law. Our relaxed formulation of the 0,1-law also permits us to discuss 0,1-laws for coherent, and even semi-coherent, classes and to make changes of language when desired – passing beyond the context of Fraïssé classes.

6.1. **Preliminaries.** In this subsection, we formulate two forms of (labeled) 0,1-law for first-order logic – Definitions 6.2 and 6.3 below. The first of these is, not surprisingly, the classic one, in which the members of a class in universe $n = \{0, 1, ..., n-1\}$ are given the uniform distribution for each n. The second is the relaxed formulation, which we call the 0,1-law*. After presenting these two formulations, we briefly sketch out some ways in which they are related.

Definition 6.2. Let $\mathbf{C}$ be a Fraïssé class in a finite relational language. For each $0 < n < \omega$, we define the set

$$\mathbf{C}[n] = \{\mathfrak{a} \in \mathbf{C} : \|\mathfrak{a}\| = \{0, 1, ..., n-1\}\}$$

and then a probability mass function $p_n^{\mathbf{C}} : \mathbf{C}[n] \to [0,1]$ is given by $p_n^{\mathbf{C}}(\mathfrak{a}) = 1/|\mathbf{C}[n]|$ for all $\mathfrak{a} \in \mathbf{C}[n]$. For a sentence $\varphi \in \mathit{Sent}(\mathscr{L})$ and $0 < n < \omega$, we define

$$\mathbb{P}_n^{\mathbf{C}}(\varphi) = \sum_{\substack{\mathfrak{a} \in \mathbf{C}[n]: \\ \mathfrak{a} \models \varphi}} p_n^{\mathbf{C}}(\mathfrak{a})$$

We define,

$$T_{\mathbf{C}}^{\mathrm{as}} = \left\{ \mathit{Sent}(\mathscr{L}) : \lim_{n \to \infty} \mathbb{P}_n^{\mathbf{C}}(\varphi) = 1 \right\}$$

which is necessarily a consistent theory, and we say that $\mathbf{C}$ has *the 0,1-law for first-order logic* just in case $T_{\mathbf{C}}^{\mathrm{as}}$ is a complete theory.[5]

Definition 6.3. Let $\mathbf{C}$ be a semi-coherent class with generic model $\mathcal{M}$. Recall that for each $\mathfrak{b} \in \mathbf{C}$, we have $\mathbf{C}(\mathfrak{b}) = \{\mathfrak{a} \in \mathbf{C} : \mathfrak{a} \leq \mathfrak{b}\}$ and for $n, s < \omega$, we define $\mathbf{C}_{n:s}(\mathfrak{b}) = \{\mathfrak{a} \in \mathbf{C}(\mathfrak{b}) : n \leq |\mathfrak{a}| \leq n + s\}$

- For $\mathfrak{b} \in \mathbf{C}(\mathcal{M})$, $n, s < \omega$, and $\varphi \in \mathit{Sent}(\mathscr{L})$, we define

$$\mathbb{P}_{n:s}(\varphi; \mathfrak{b}) = \begin{cases} \frac{\left|\{\mathfrak{a} \in \mathbf{C}_{n:s}(\mathfrak{b}) : \mathfrak{a} \models \varphi\}\right|}{\left|\mathbf{C}_{n:s}(\mathfrak{b})\right|} & \text{if } \mathbf{C}_{n:s}(\mathfrak{b}) \neq \emptyset \\ 0 & \text{if } \mathbf{C}_{n:s}(\mathfrak{b}) = \emptyset \end{cases}$$

- We write $(\mathfrak{b}_k)_k \to \mathcal{M}$ to indicate that $(\mathfrak{b}_k)_k$ is a $\leq$-chain of members of $\mathbf{C}(\mathcal{M})$ such that $\bigcup_k \mathfrak{b}_k = \mathcal{M}$ and $|\mathfrak{b}_k| > k^2$, and if $(\mathfrak{b}_k)_k \to \mathcal{M}$ and $\varphi \in \mathit{Sent}(\mathscr{L})$, then we define,

$$\mathbb{P}_\infty(\varphi; (\mathfrak{b}_k)_k) = \lim_{n \to \infty} \left(\liminf_{k \to \infty} \mathbb{P}_{n:k}(\varphi; \mathfrak{b}_k) \right).$$

We say that $\mathbf{C}$ *has the 0,1-law* for first-order logic* if one can find $(\mathfrak{b}_k)_k \to \mathcal{M}$ such that

$$T^*(\mathfrak{b}_k)_k = \{\varphi \in \mathit{Sent}(\mathscr{L}) : \mathbb{P}_\infty(\varphi; (\mathfrak{b}_k)_k) = 1\}$$

is a complete theory.

[5]In fact, we should say that $\mathbf{C}$ has the *labeled 0,1-law for first-order logic under uniform distributions*, but that phrase is a bit cumbersome. We use a similar abbreviation for the other kind of 0,1-law in play – the 0,1-law*.

Next, we'll suggest two connections between our two forms of 0,1-law. It appears that this list is far from exhaustive, but producing such a list would amount to just a large amount of more-or-less direct calculation. As this would take us far from the usual context of a paper on logic, we will leave such calculating for another day.[6] The subsequent observation shows that "heavy lifting" machinery underlying Theorem 6.1 has implications for "sparse" 0,1-laws (which pertain to abstract strictly-semi-coherent classes) as well.

Lemma 6.4. *Suppose* $\mathbf{C}$ *is a Fraïssé class in a finite relational language with trivial algebraic closure. If* $\mathbf{C}$ *has the 0,1-law for first-order logic with* $T^{\mathrm{as}}_{\mathbf{C}} = T_{\mathbf{C}}$*, then* $\mathbf{C}$ *has the 0,1-law* for first-order logic with* $T^*(\mathfrak{b}_k)_k = T^{\mathrm{as}}_{\mathbf{C}} = T_{\mathbf{C}}$.

Proof (sketch). One just observes that

$$\frac{\left|\{\mathfrak{a} \in \mathbf{C}_{n:s}(\mathfrak{b}) : \mathfrak{a} \vDash \varphi\}\right|}{\left|\mathbf{C}_{n:s}(\mathfrak{b})\right|} = \frac{\sum_{r=n}^{n+s} \binom{n+s}{r} \left|\{\mathfrak{a} \in \mathbf{C}(\mathfrak{b}) : |\mathfrak{a}| = r \wedge \mathfrak{a} \vDash \varphi\}\right|}{\sum_{k=n}^{n+s} \binom{n+s}{r} \left|\{\mathfrak{a} \in \mathbf{C}(\mathfrak{b}) : |\mathfrak{a}| = r\}\right|}$$

and that for all $m < n < \omega$ and $\mathfrak{a} \in \mathbf{C}[m]$,

$$\left|\mathbf{C}[m]\right|^{-1} = \left|\{\mathfrak{b} \in \mathbf{C}[n] : \mathfrak{a} \leq \mathfrak{b}\}\right| \cdot \left|\mathbf{C}[n]\right|^{-1}.$$

□

Observation 6.5. Let $\mathscr{L}$ be a finite relational language, and let $\alpha = (\alpha_R : R \in \mathrm{sig}(\mathscr{L}))$ be a family of irrational numbers in $(0, 1)$. Define $\delta_\alpha : \mathrm{Fin}^\sim(\mathscr{L}) \to \mathbb{R}$ by,

$$\delta_\alpha(\mathfrak{a}) = |\mathfrak{a}| - \sum_{R^{(n)} \in \mathrm{sig}(\mathscr{L})} \frac{\alpha_R}{n!} \left|R^{\mathfrak{a}}\right|$$

where $\mathrm{Fin}^\sim(\mathscr{L})$ is the class of finite $\mathscr{L}$-structures $\mathfrak{a}$ such that for each $R^{(n)} \in \mathrm{sig}(\mathscr{L})$, if $\overline{a} \in R^{\mathfrak{a}}$, then $a_i \neq a_j$ whenever $i < j < n$, and $(a_{\sigma(0)}, ..., a_{\sigma(n-1)}) \in R^{\mathfrak{a}}$ for every $\sigma \in Sym(n)$. Let $\mathbf{C}_\alpha = \{\mathfrak{a} : \delta_\alpha(\mathfrak{a}) \geq 0\}$. For $\mathfrak{a} \in \mathbf{C}_\alpha$, we define $d^{\mathfrak{a}}_\alpha : \mathcal{P}(\|\mathfrak{a}\|) \to \mathbb{R}$ by

$$d^{\mathfrak{a}}_\alpha(X) = \inf\{\delta_\alpha(\mathfrak{a}[B]) : X \subseteq B \subseteq \|\mathfrak{a}\|\}$$

and then we define a strong-substructure relation $\leq_\alpha$ by

$$\mathfrak{a} \leq_\alpha \mathfrak{b} \;\Leftrightarrow\; \left[\mathfrak{a} \leq \mathfrak{b} \;\&\; d^{\mathfrak{a}}_\alpha = d^{\mathfrak{b}}_\alpha \restriction \mathcal{P}(\|\mathfrak{a}\|)\right].$$

Then $(\mathbf{C}_\alpha, \leq_\alpha)$ is an abstract semi-coherent class. Let $\tilde{\mathbf{C}}_\alpha$ be the semi-coherent class derived from $(\mathbf{C}_\alpha, \leq_\alpha)$ as in Definition 2.16. Then $\tilde{\mathbf{C}}_\alpha$ has the 0,1-law* for first-order logic with $T^*(\mathfrak{b}_k)_k = T_{\mathbf{C}}$ for appropriately chosen $(\mathfrak{b}_k)_k \to \mathcal{M}$.

6.2. **Proof of Theorem 6.1.** At last, we turn to the proof of Theorem 6.1. In fact, we will recover Theorem 6.1 as an immediate corollary of the following proposition and Proposition 3.11.

Proposition 6.6. *Let* $\mathbf{C}$ *be a semi-coherent class with generic model* $\mathcal{M}$*. Suppose* $\mathbf{C}$ *has the 0,1-law* via* $(\mathfrak{b}_k)_k \to \mathcal{M}$ *such that* $T^*(\mathfrak{b}_k)_k = T_{\mathbf{C}}$*. Then* $\mathbf{C}$ *is semi-tame.*

Definition 6.7. Let $\mathbf{C}$ be a semi-coherent class with generic model $\mathcal{M}$, and let $(\mathfrak{b}_k)_k \to \mathcal{M}$. We define $s : \mathbf{C}(\mathcal{M}) \times \omega \to \omega$ by setting

$$s(\mathfrak{b}, n) = \max\{0, \max\{|\mathfrak{b}_k| - n : \mathfrak{b}_k \leq^* \mathfrak{b}\}\}.$$

Now, for $F \subset_{\mathrm{fin}} Sent(\mathscr{L})$, $\varepsilon > 0$ and $0 < n < \omega$, we define

$$D(F, \varepsilon, n; (\mathfrak{b}_k)_k) = \left\{\mathfrak{b} \in \mathbf{C}(\mathcal{M}) : |\mathfrak{b}| \geq n + s(\mathfrak{b}, n) \wedge \mathbb{P}_{n:s(\mathfrak{b})}(\wedge F; \mathfrak{b}) \geq 1 - \varepsilon\right\}.$$

Also, for $r < \omega$, we have $Z_r(\mathcal{M}) = \left\{\mathfrak{a} \in \mathbf{C}(\mathcal{M}) : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq r\right\}$

[6]We should also note that the restriction of Theorem 6.1 is due just to the fact that here, we haven't gone beyond Lemma 6.4.

Lemma 6.8. *Let $\mathbf{C}$ be a coherent class with generic model $\mathcal{M}$. Suppose $\mathbf{C}$ has the 0,1-law* for first-order logic and $T^*(\mathfrak{b}_k)_k = T_{\mathbf{C}}$. There is a function $H : T_{\mathbf{C}} \times (0,1) \to \omega$ such that*

$$\left\{ X \subseteq \mathbf{C}(\mathcal{M}) : \begin{array}{c} (\exists\, F \subset_{\mathit{fin}} T_{\mathbf{C}},\, \varepsilon > 0,\, \mathfrak{a} \in \mathbf{C}(\mathcal{M}))\, (\forall n, r) \\ \qquad H(F,\varepsilon) {\leq} n \;\Rightarrow\; \mathbf{C}_{\mathfrak{a}}(\mathcal{M}) \cap D\left(F, \varepsilon, n; (\mathfrak{b}_k)_k\right) \cap Z_r(\mathcal{M}) \subseteq X \end{array} \right\}$$

is a (proper) filter on $\mathbf{C}(\mathcal{M})$ extending $\mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathcal{M})$.

Proof. It follows immediately from the hypotheses that the function $H : \mathcal{P}_{\mathrm{fin}}(T_{\mathbf{C}}) \times (0,1) \to \omega$ given by

$$H(F,\varepsilon) = \min\left\{ n : (\forall \mathfrak{a} \in \mathbf{C}(\mathcal{M}))\, \mathbf{C}_{\mathfrak{a}}(\mathcal{M}) \cap D\left(F, \varepsilon, n; (\mathfrak{b}_k)_k\right) \neq \emptyset \right\}$$

is well-defined. Furthermore, if $H(F,\varepsilon) \leq n$, then $\mathbf{C}_{\mathfrak{a}}(\mathcal{M}) \cap D(F,\varepsilon,n) \cap Z_r(\mathcal{M}) \neq \emptyset$ as well. (From now on, we use the notation $D(F,\varepsilon,n)$ in place of the more cumbersome $D\left(F,\varepsilon,n;(\mathfrak{b}_k)_k\right)$.)

Now, to verify the finite intersection property, let $X, Y \subseteq \mathbf{C}(\mathcal{M})$ such that there are $F_X, F_Y \subset_{\mathrm{fin}} T_{\mathbf{C}}$, $0 < \varepsilon_X, \varepsilon_Y < 1$ and $\mathfrak{a}_X, \mathfrak{a}_Y \in \mathbf{C}(\mathcal{M})$ such that

$$H(F_X,\varepsilon_X) {\leq} n \Rightarrow \mathbf{C}_{\mathfrak{a}_X}(\mathcal{M}) {\cap} D\big(F_X, \varepsilon_X, n\big) {\cap} Z_r(\mathcal{M}) \subseteq X$$
$$H(F_Y,\varepsilon_Y) {\leq} n \Rightarrow \mathbf{C}_{\mathfrak{a}_Y}(\mathcal{M}) {\cap} D\big(F_Y, \varepsilon_Y, n\big) {\cap} Z_r(\mathcal{M}) \subseteq Y$$

for all n and r. We choose $\mathfrak{b} \in \mathbf{C}_{\mathfrak{a}_X}(\mathcal{M}) \cap \mathbf{C}_{\mathfrak{a}_Y}(\mathcal{M})$ and $\varepsilon = \min\{\varepsilon_X, \varepsilon_Y\}$. It is clear, then, that if

$$\max\left\{ H(F_X,\varepsilon_X), H(F_Y,\varepsilon_Y) \right\} \leq H(F_X {\cup} F_Y, \varepsilon) \leq n$$

then for every r,

$$\begin{aligned} \mathbf{C}_{\mathfrak{b}}(\mathcal{M}) \cap D(F_X {\cup} F_Y, \varepsilon, n) \cap Z_r(\mathcal{M}) &\subseteq \bigcap \begin{cases} \mathbf{C}_{\mathfrak{b}}(\mathcal{M}) \cap D(F_X, \varepsilon_X, n) \cap Z_r(\mathcal{M}) \\ \mathbf{C}_{\mathfrak{b}}(\mathcal{M}) \cap D(F_Y, \varepsilon_Y, n) \cap Z_r(\mathcal{M}) \end{cases} \\ &\subseteq \bigcap \begin{cases} \mathbf{C}_{\mathfrak{a}_X}(\mathcal{M}) \cap D(F_X, \varepsilon_X, n) \cap Z_r(\mathcal{M}) \\ \mathbf{C}_{\mathfrak{a}_Y}(\mathcal{M}) \cap D(F_Y, \varepsilon_Y, n) \cap Z_r(\mathcal{M}) \end{cases} \\ &\subseteq X \cap Y. \end{aligned}$$

□

Proof of Proposition 6.6. To prove the theorem, we must "pare down" the original class $\mathbf{C}$, recovering a cofinal super-robust subclass. This process is encoded in the following construction:

Construction: We enumerate $T_{\mathbf{C}} = \{\varphi_0, ..., \varphi_n, ...\}$, and for each $n < \omega$, we set $F_n = \{\varphi_i\}_{i \leq n}$. We define a sequence of classes

$$\mathbf{C}^*_0 \supseteq \mathbf{C}^*_1 \supseteq \cdots \supseteq \mathbf{C}^*_n \supseteq \cdots$$

of length ω as follows:

- To start with, we just set $\mathbf{C}^*_0 = \mathbf{C}$ and $k_0 = 0$.
- At the beginning of stage $n+1$, we are given the class $\mathbf{C}^*_n$. Setting

  $$k_{n+1} = \max\left\{ k_n + n + 2,\, \max_{m \leq n} H(F_m, 2^{-n}) \right\}$$

  we select $\mathbf{C}^*_{n+1} \subseteq \mathbf{C}^*_n$ so that:
  - $\mathbf{C}^*_{n+1}[{\geq} k_{n+1}] \cap \mathbf{C}(\mathcal{M}) = D\left(F_n, 2^{-n}, k_{n+1}\right) \cap \mathbf{C}^*_n$ where
  - $\mathbf{C}^*_{n+1}[{<} k_{n+1}] = \mathbf{C}^*_n[{<} k_{n+1}]$

  where $\mathbf{C}^*_{n+1}[{\geq} k_{n+1}] := \bigcup_{k_{n+1} \leq m} \mathbf{C}^*_{n+1}[m]$ and $\mathbf{C}^*_{n+1}[{<} k_{n+1}] := \bigcup_{m < k_{n+1}} \mathbf{C}^*_{n+1}[m]$

Finally, we have a defined a class $\mathbf{C}^* = \bigcap_{n < \omega} \mathbf{C}^*_n$. This completes the construction.

Claim. $\mathbf{C}^*$ is a semi-coherent cofinal subclass of $\mathbf{C}$.

Proof of claim. AP and JEP problems and their solutions (for members of $\mathbf{C}$) are encoded in $T_{\mathbf{C}}$, and semi-coherence follows immediately from this fact. For cofinality in $\mathbf{C}$, it is enough to observe that for every $\mathfrak{a} \in \mathbf{C}$, there is a sentence $\varphi_{\mathfrak{a}} \in T_{\mathbf{C}}$ such that for any $\mathscr{L}$-structure $\mathcal{N}$, $\mathcal{N} \vDash \varphi_{\mathfrak{a}}$ if and only if there is an embedding $\mathfrak{a} \to \mathcal{N}$; by construction every $\mathfrak{b} \in \mathbf{C}^*$ of sufficiently large rank satisfies $\varphi_{\mathfrak{a}}$. □

Claim. $\mathbf{C}^*$ is super-robust.

Proof of claim. It is clear that $\mathbf{C}^*$ is well-quasi-ordered by $\leq^*$ up to foundation rank, so to apply Proposition 3.11, we just need to verify that $\mathbf{C}^*$ has perfect-FSP up to foundation rank. Let $\mathscr{U}$ be an ultrafilter on $\mathbf{C}^*(\mathcal{M})$ extending $\mathsf{Cone}^{\infty}_{\mathbf{C}^*}(\mathcal{M})$; as $\mathscr{U}$ is arbitrary, by Theorem 3.8, it is enough for us to show that $w^{\mathscr{U}} : \mathcal{M} \to \Pi\mathbf{C}^*(\mathcal{M})/\mathscr{U}$ is an elementary embedding. Let $\varphi(x_0, ..., x_{s-1}) \in \mathscr{L}$ and $\bar{b} \in M^s$ be given; we must show that $\mathcal{M} \vDash \varphi(\bar{b}) \Leftrightarrow \Pi\mathbf{C}^*(\mathcal{M})/\mathscr{U} \vDash \varphi(w^{\mathscr{U}}\bar{b})$. As $T_{\mathbf{C}}$ is model-complete and $T_{\mathbf{C}} = Th\left(\Pi\mathbf{C}^*(\mathcal{M})/\mathscr{U}\right)$, to verify that $\mathcal{M} \vDash \varphi(\bar{b}) \Leftrightarrow \Pi\mathbf{C}^*(\mathcal{M})/\mathscr{U} \vDash \varphi(w^{\mathscr{U}}\bar{b})$ for all $\varphi \in \mathscr{L}$, it will suffice to verify that

$$\mathcal{M} \vDash \exists\bar{y}\,\theta(\bar{b}, \bar{y}) \Leftrightarrow \Pi\mathbf{C}^*(\mathcal{M})/\mathscr{U} \vDash \exists\bar{y}\,\theta(w^{\mathscr{U}}\bar{b}, \bar{y}).$$

whenever $\theta(\bar{x}, , y_0, ..., y_{t-1})$ is a quantifier-free formula. Since θ is quantifier-free, the proof of $\Rightarrow$ is trivial. For $\Leftarrow$, suppose $\Pi\mathbf{C}^*(\mathcal{M})/\mathscr{U} \vDash \theta(w^{\mathscr{U}}\bar{b}, \bar{f}/\mathscr{U})$ for some $f_0, ..., f_{t-1} \in \Pi\mathbf{C}^*(\mathcal{M})$ – that is,

$$Y = \left\{\mathfrak{a} \in \mathbf{C}^*(\mathcal{M}) : \mathfrak{a} \vDash \theta\left(\bar{b}, \bar{f}(\mathfrak{a})\right)\right\} \in \mathscr{U}.$$

Now, let $\mathfrak{b} \in \mathbf{C}^*(\mathcal{M})$ such that $\bar{b} \subseteq \|\mathfrak{b}\|$. Then,

$$X \cap Y \cap \mathbf{C}^*_{\mathfrak{b}}(\mathcal{M}) \in \mathscr{U},$$

and since $\mathscr{U}$ is a proper filter, we may choose some $\mathfrak{c}$ in this set; in particular, $\mathfrak{c} \vDash \theta(\bar{b}, \bar{f}(\mathfrak{c}))$. Once again because θ is quantifier-free, we have $\mathcal{M} \vDash \theta(\bar{b}, \bar{f}(\mathfrak{c}))$, as required. □

This completes the proof of Theorem 6.1. □

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Appendix A. Proofs of statements from Section 3

A.1. **Proof of the Theorem 3.8.** In this subsection, obviously, we present the proof of Theorem 3.8. Rather than include a lot of interstitial text, we expose the outline of the argument in the following table.

1 ⇒ 2,3,4	Lemmas A.5 and A.6, and Corollary A.4
4 ⇒ 3	Lemma A.7
3 ⇒ 2	Lemma A.8
2 ⇒ 1	Lemma A.9

Lemma A.1. *Let* $\mathbf{C}$ *be a super-robust semi-coherent class (via* $\nu : \mathscr{L} \to \omega$*), and let* $\mathcal{M}$ *be its generic model. Then for every formula* $\varphi(x_0, ..., x_{k-1}) \in \mathscr{L}$*, there is a number* $e_\varphi < \omega$ *such that for every* $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$ *and* $\bar{b} \in \|\mathfrak{a}\|^k$*, if* $\mathrm{rk}(\mathfrak{a}) \geq e_\varphi$*, then* $\mathfrak{a} \vDash \varphi(\bar{b})$ *iff* $\mathcal{M} \vDash \varphi(\bar{b})$*.*

Proof. We proceed by induction on formulas.

- If $\varphi(x_0, ..., x_{k-1})$ is quantifier-free, then we can take $e_\varphi = 0$.
- For the boolean connectives, we can take $e_{\neg\varphi} = e_\varphi$ and $e_{\varphi\wedge\psi} = \max\{e_\varphi, e_\psi\}$.
- ($\forall$) Suppose we are given an $e_\varphi < \omega$ where $\varphi = \varphi(x_0, ..., x_{k-1}, y)$. Then, we choose

$$e = e_{\forall y\, \varphi} = \max\left\{e_\varphi, \nu\big(\forall y\, \varphi(\overline{x}, y)\big)\right\}.$$

To see that this choice works, let $\mathfrak{a} \in \mathbf{C}(\mathcal{M})$ and $\bar{b} \in \|\mathfrak{a}\|^k$ be given, and assume $\mathrm{rk}(\mathfrak{a}) \geq e$.

First, suppose $\mathcal{M} \vDash \forall y\, \varphi(\bar{b}, y)$; then, let $c \in \|\mathfrak{a}\|$ be given. Since $\mathrm{rk}(\mathfrak{a}) \geq e \geq e_\varphi$, we know that $\mathfrak{a} \vDash \varphi(\bar{b}, c)$ iff $\mathcal{M} \vDash \varphi(\bar{b}, c)$, so $\mathfrak{a} \vDash \varphi(\bar{b}, c)$. As c was arbitrary, we have shown that $\mathfrak{a} \vDash \forall y\, \varphi(\bar{b}, y)$.

Now, suppose $\mathfrak{a} \vDash \forall y\, \varphi(\bar{b}, y)$; then let $c \in M$ be given – we must show that $\mathcal{M} \vDash \varphi(\bar{b}, c)$. Since $\mathcal{M}$ is the generic model of $\mathbf{C}$, we may choose $\mathfrak{a}' \in \mathbf{C}(\mathcal{M})$ such that $\mathfrak{a} \leq \mathfrak{a}'$ and $c \in \|\mathfrak{a}'\|$. Since $\mathrm{rk}(\mathfrak{a}') \geq \mathrm{rk}(\mathfrak{a}) \geq e_\varphi$, we know that $\mathcal{M} \vDash \varphi(\bar{b}, c)$ iff $\mathfrak{a}' \vDash \varphi(\bar{b}, c)$. Further, since $\mathrm{rk}(\mathfrak{a}) \geq \nu(\forall y\, \varphi)$ and $\mathfrak{a} \vDash \forall y\, \varphi(\bar{b}, y)$, we know that $\mathfrak{a}' \vDash \forall y\, \varphi(\bar{b}, y)$, and in particular, $\mathfrak{a}' \vDash \varphi(\bar{b}, c)$. Thus, $\mathcal{M} \vDash \varphi(\bar{b}, c)$, as desired.

This completes the proof. □

Lemma A.2. *Let* $\mathbf{C}$ *be a robust semi-coherent class. Then,*

$$T^\infty_{\mathbf{C}} = \{\varphi \in \mathit{Sent}(\mathscr{L}) : \{\mathrm{rk}(\mathfrak{a}) : \mathfrak{a} \in \mathbf{C},\, \mathfrak{a} \vDash \varphi\} \text{ is infinite}\}$$

is a consistent complete theory.

Proof. Assume $\mathbf{C}$ is robust via $\nu : \mathscr{L} \times \omega \to \omega$. For completeness, let $\varphi \in \mathit{Sent}(\mathscr{L})$ be given, and suppose $\{\mathrm{rk}(\mathfrak{a}) : \mathfrak{a} \in \mathbf{C}, \mathfrak{a} \vDash \varphi\}$ is finite. Then, clearly, $\{\mathrm{rk}(\mathfrak{a}) : \mathfrak{a} \in \mathbf{C}, \mathfrak{a} \vDash \neg\varphi\}$ is infinite, so $\neg\varphi \in T^\infty_{\mathbf{C}}$.

Now, we consider consistency. By induction on $n = 1, 2, 3, ...$, we show that if $\varphi_0, ..., \varphi_{n-1} \in T^\infty_{\mathbf{C}}$, $\{\varphi_0, ..., \varphi_n\}$ is satisfiable. When $n = 1$, there is nothing to do, so we move to the induction step. Let $\varphi_0, ..., \varphi_{n-1}, \varphi_n \in T^\infty_{\mathbf{C}}$ be given. Let $X = \left\{\mathrm{rk}(\mathfrak{a}) : \mathfrak{a} \vDash \bigwedge_{i<n} \varphi_i\right\}$ and $Y = \{\mathrm{rk}(\mathfrak{a}) : \mathfrak{a} \vDash \varphi_n\}$, both of which are infinite.

Let $r = \max_{i \leq n} \nu(\varphi_i, 0)$. Since X is infinite, we can choose $\mathfrak{a}_X \in \mathbf{C}$ such that $\mathfrak{a}_X \vDash \bigwedge_{i<n} \varphi_i$ and $\mathrm{rk}(\mathfrak{a}_X) \geq r$. Similarly, we can choose $\mathfrak{a}_Y \in \mathbf{C}$ such that $\mathfrak{a}_Y \vDash \varphi_n$ and $\mathrm{rk}(\mathfrak{a}_Y) \geq r$. We may also choose $\mathfrak{a}^0_X \leq \mathfrak{a}_X$, $\mathfrak{a}^0_Y \leq \mathfrak{a}_Y$, members of $\mathbf{C}$ of foundation rank 0. Finally, let $\mathfrak{b} \in \mathbf{C}$ such that there are embeddings $\mathfrak{a}_X \to \mathfrak{b}$ and $\mathfrak{a}_Y \to \mathfrak{b}$. Then we have

$$\begin{aligned} \mathrm{rk}(\mathfrak{a}_X/\mathfrak{a}^0_X) &\geq \max\nolimits_{i<n} \nu(\varphi_i, 0), \quad \text{so } \mathfrak{b} \vDash \textstyle\bigwedge_{i<n} \varphi_i \\ \mathrm{rk}(\mathfrak{a}_Y/\mathfrak{a}^0_Y) &\geq \nu(\varphi_n, 0), \quad \text{so } \mathfrak{b} \vDash \varphi_n \end{aligned}$$

and so $\mathfrak{b}$ shows that $\{\varphi_0, ..., \varphi_{n-1}, \varphi_n\}$ is consistent. $\square$

Corollary A.3. *Let $\mathbf{C}$ be a robust semi-coherent class. Then $\mathbf{C}$ has the following universality property: For every $\mathfrak{a} \in \mathbf{C}$, there is a number $r_{\mathfrak{a}} < \omega$ such that for all $\mathfrak{b} \in \mathbf{C}$, if $\mathrm{rk}(\mathfrak{b}) \geq r_{\mathfrak{a}}$, then $\mathfrak{a} \leq^* \mathfrak{b}$.*

Proof. Given $\mathfrak{a} \in \mathbf{C}$, it is easy to see that there is a model $\mathscr{N} \vDash T^{\infty}_{\mathbf{C}}$ such that $\mathfrak{a} \leq^* \mathscr{N}$. Then one applies Lemma A.2 to complete the proof. $\square$

Corollary A.4. *[1⇒3]*

Proof. By Lemmas A.1 and A.2. $\square$

Lemma A.5. *[1⇒2]*

Proof. To begin with, we prove item 2 without the expansion. Let $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ be a sequence of members of $\mathbf{C}$ such that the set $\{\mathrm{rk}(\mathfrak{a}_n) : n < \omega\}$ is infinite. Fix an enumeration $a_0, ..., a_{k-1}$ of $\|\mathfrak{a}_0\|$, and let $\theta(x_0, ..., x_{k-1})$ be the quantifier-free-complete type of $(a_0, ..., a_{k-1})$ in $\mathfrak{a}_0$, and let $\varphi = \exists \overline{x}\, \theta(\overline{x})$. Since $\mathscr{M}$ is the generic model of $\mathbf{C}$, we know that $\mathscr{M} \vDash \varphi$. Using the notation of Lemma A.1 and using the hypothesis on the sequence, we may select $0 < n < \omega$ such that $\mathrm{rk}(\mathfrak{a}_n) \geq e_{\varphi}$; then $\mathfrak{a}_n \vDash \varphi$, $\mathfrak{a}_0 \leq^* \mathfrak{a}_n$.

To extend this argument to expansions $\mathbf{C}_{\mathfrak{a}}$, one needs only to observe that the quantifier-free-complete type of any $\mathfrak{a} \in \mathbf{C}(\mathscr{M})$ isolates its complete type. $\square$

Lemma A.6. *[1⇒4]*

Proof. Assuming $\mathbf{C}$ is super-robust via $\nu : \mathscr{L} \to \omega$, let $\mathscr{U}$ be a non-principal ultrafilter on $\mathbf{C}(\mathscr{M})$ such that $\mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathscr{M}) \subseteq \mathscr{U}$. It is clear that for every $e < \omega$, the set $X_e = \{\mathfrak{a} \in \mathbf{C}(\mathscr{M}) : \mathrm{rk}(\mathfrak{a}) \geq e\}$ is in $\mathscr{U}$, so as $\mathbf{C}$ is super-robust and by Lemma A.1, we know that $\Pi\mathbf{C}(\mathscr{M})/\mathscr{U} \vDash T_{\mathbf{C}}$.

Let $\varphi(x_0, ..., x_{k-1}) \in \mathscr{L}$ and $\overline{b} \in M^k$ be given. Since $T_{\mathbf{C}}$ is model-complete, there are a quantifier-free formula $\theta_{\varphi}(\overline{x}, y_0, ..., y_{\ell-1}) \in \mathscr{L}$ and a finite subset $F \subset_{\mathrm{fin}} T_{\mathbf{C}}$ such that $F \vDash \varphi(\overline{x}) \leftrightarrow \exists \overline{y}\, \theta_{\varphi}(\overline{x}, \overline{y})$. Setting $e_F = \max_{\psi \in F} e_{\psi}$ in the notation of Lemma A.1, we see that (i) for every $\mathfrak{a} \in X_{e_F}$, $\mathfrak{a} \vDash \varphi(\overline{x}) \leftrightarrow \exists \overline{y}\, \theta_{\varphi}(\overline{x}, \overline{y})$ and (ii) $\Pi\mathbf{C}(\mathscr{M})/\mathscr{U} \vDash \varphi(\overline{x}) \leftrightarrow \exists \overline{y}\, \theta_{\varphi}(\overline{x}, \overline{y})$. Thus, it will suffice show that $\mathscr{M} \vDash \exists \overline{y}\, \theta_{\varphi}(\overline{b}, \overline{y})$ if and only if $\Pi\mathbf{C}(\mathscr{M})/\mathscr{U} \vDash \exists \overline{y}\, \theta_{\varphi}(w^{\mathscr{U}}(b_0), ..., w^{\mathscr{U}}(b_{k-1}), \overline{y})$.

First, we suppose $\mathscr{M} \vDash \exists \overline{y}\, \theta_{\varphi}(\overline{b}, \overline{y})$. In particular, let $\overline{c} \in M^{\ell}$ such that $\mathscr{M} \vDash \theta_{\varphi}(\overline{b}, \overline{c})$. Let $e = \max\{e_F, e_{\theta_{\varphi}}\}$, and let $\mathfrak{a}_0 \in X_e$ such that $\overline{b}\overline{c} \subseteq \|\mathfrak{a}_0\|$. For every $\mathfrak{a} \in \mathbf{C}_{\mathfrak{a}_0}(\mathscr{M})$, we have $\mathrm{rk}(\mathfrak{a}) \geq e$, so $\mathfrak{a} \vDash \theta_{\varphi}(\overline{b}, \overline{c})$, and so

$$\mathbf{C}_{\mathfrak{a}}(\mathscr{M}) \cap X_e \subseteq Y(\theta_{\varphi}, \overline{b}\overline{c}) = \left\{\mathfrak{a} \in \mathbf{C}(\mathscr{M}) : \mathfrak{a} \vDash \theta_{\varphi}(w_{b_0}(\mathfrak{a}), ...w_{b_{k-1}}(\mathfrak{a}), w_{c_0}(\mathfrak{a}), ..., w_{c_{\ell-1}}(\mathfrak{a}))\right\}$$

for every $e < \omega$, showing that $Y(\theta_{\varphi}, \overline{b}\overline{c}) \in \mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathscr{M}) \subseteq \mathscr{U}$. Of course, this means that

$$\Pi\mathbf{C}(\mathscr{M})/\mathscr{U} \vDash \theta_{\varphi}(w^{\mathscr{U}}(b_0), ..., w^{\mathscr{U}}(b_{k-1}), w^{\mathscr{U}}(c_0), ..., w^{\mathscr{U}}(c_{\ell-1}))$$

so $\Pi\mathbf{C}(\mathscr{M})/\mathscr{U} \vDash \exists \overline{y}\, \theta_{\varphi}(w^{\mathscr{U}}\overline{b}, \overline{y})$, as required.

Conversely, suppose $\Pi\mathbf{C}(\mathscr{M})/\mathscr{U} \vDash \exists \overline{y}\, \theta_{\varphi}(w^{\mathscr{U}}\overline{b}, \overline{y})$. In particular, let $\xi_0, ..., \xi_{\ell-1} \in \Pi\mathbf{C}(\mathscr{M})$ and $Y \in \mathscr{U}$ such that

$$Z \subseteq \left\{\mathfrak{a} \in \mathbf{C}(\mathscr{M}) : \mathfrak{a} \vDash \theta_{\varphi}(w_{b_0}(\mathfrak{a}), ...w_{b_{k-1}}(\mathfrak{a}), \xi_0(\mathfrak{a}), ..., \xi_{\ell-1}(\mathfrak{a}))\right\}.$$

Towards a contradiction, suppose $\mathscr{M} \nvDash \exists \overline{y}\, \theta_{\varphi}(\overline{b}, \overline{y})$. Again, we take $e = \max\{e_F, e_{\theta_{\varphi}}\}$. Then since $\mathscr{U}$ is a filter, $X_e \cap Z \in \mathscr{U}$, and in particular, $X_e \cap Z$ is non-empty; let $\mathfrak{a} \in X_e \cap Z$. Since $\mathrm{rk}(\mathfrak{a}) \geq e$, $\mathfrak{a} \vDash \neg\exists \overline{y}\, \theta_{\varphi}(\overline{b}, \overline{y})$; but since $\mathfrak{a} \in Z$, $\mathfrak{a} \vDash \theta_{\varphi}(\overline{b}, \overline{\xi}(\mathfrak{a}))$, so $\mathfrak{a} \vDash \exists \overline{y}\, \theta_{\varphi}(\overline{b}, \overline{y})$ – a contradiction. Thus, $\mathscr{M} \vDash \exists \overline{y}\, \theta_{\varphi}(\overline{b}, \overline{y})$, as desired. This completes the proof of 1⇒4. $\square$

Lemma A.7 (4⇒3).

Proof. Assuming perfect-FSP/rank, let $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ be an enumeration of the isomorphism types in $\mathbf{C}$. Also, let $\mathscr{U}_0$ be a non-principal ultrafilter on ω such that $\{n : \mathrm{rk}(\mathfrak{a}_n) \geq e\} \in \mathscr{U}$ for every $e < \omega$. We define a family $\mathscr{U}_0'$ of a subsets of $\mathbf{C}(\mathcal{M})$ as,

$$\left\{X \subseteq \mathbf{C}(\mathcal{M}) : \big(\exists Y{\in}\mathscr{U}_0,\, \mathfrak{b}{\in}\mathbf{C}(\mathcal{M})\big)\, \mathbf{C}_{\mathfrak{b}}(\mathcal{M}) \cap \{\mathfrak{a}{\in}\mathbf{C}(\mathcal{M}) : \mathfrak{a} \cong \mathfrak{a}_n,\, n{\in}Y\} \subseteq X\right\}.$$

The fact that $\mathscr{U}_0'$ is a proper filter on $\mathbf{C}(\mathcal{M})$ follows from Corollary A.3. Let $\mathscr{V}$ be an ultrafilter on $\mathbf{C}(\mathcal{M})$ such that $\mathsf{Cone}^{\infty}_{\mathbf{C}}(\mathcal{M}) \subseteq \mathscr{U}_0' \subseteq \mathscr{V}$. We also define,

$$\mathscr{V}^{\bullet} = \{Y \subseteq \omega : (\exists\, X \in \mathscr{V})\, X^{\bullet} \subseteq Y\}$$

where $X^{\bullet} = \{n : \mathfrak{a}_n \cong \mathfrak{a} \in X\}$ for each $X \in \mathscr{U}$.

Claim. For any sentence $\varphi \in \mathit{Sent}(\mathscr{L})$, $\prod_n \mathfrak{a}_n/\mathscr{U} \vDash \varphi \Leftrightarrow \Pi\mathbf{C}(\mathcal{M})/\mathscr{V} \vDash \varphi$.

Proof of claim. If $Y_\varphi = \{\mathfrak{a} \in \mathbf{C}(\mathcal{M}) : \mathfrak{a} \vDash \varphi\}$ is in $\mathscr{V}$, then $\{\mathrm{rk}(\mathfrak{a}) : \mathfrak{a} \in \mathbf{C},\, \mathfrak{a} \vDash \varphi\}$ is infinite – so $\varphi \in T^{\infty}_{\mathbf{C}}$. Since $\prod_n \mathfrak{a}_n/\mathscr{U} \vDash T^{\infty}_{\mathbf{C}}$, it follows that $\prod_n \mathfrak{a}_n/\mathscr{U} \vDash \varphi$. The claim follows form the fact that $Th(\Pi\mathbf{C}(\mathcal{M})/\mathscr{V})$ and $T^{\infty}_{\mathbf{C}}$ are complete theories. □

To conclude the argument: By perfect-FSP/rank, $w^{\mathscr{V}} : \mathcal{M} \to \Pi\mathbf{C}(\mathcal{M})/\mathscr{V}$ is an elementary embedding, so in particular, we have

$$\mathcal{M} \equiv \Pi\mathbf{C}(\mathcal{M})/\mathscr{V} \equiv \Pi_n\mathfrak{a}_n/\mathscr{U}$$

and this completes the proof. □

Lemma A.8. *[3⇒2]*

Proof. Suppose $\mathbf{C}$ is generically-categorical up to foundation rank. Let $\mathfrak{a}_0, ..., \mathfrak{a}_n, ...$ be some list of members of $\mathbf{C}$ such that $\{\mathrm{rk}(\mathfrak{a}_n) : n < \omega\}$ is infinite, and without loss of generality, assume that the $\mathfrak{a}_n$'s are pairwise non-isomorphic. Let $\mathfrak{b}_0, ..., \mathfrak{b}_k, ...$ be a complete list of isomorphism types in $\mathbf{C}$, and $f : \omega \to \omega$ be the unique injection such that $\mathfrak{a}_n \cong \mathfrak{b}_{f(n)}$ for all $n < \omega$. Finally, let θ be an existential sentence of $\mathscr{L}$ such that for any $\mathscr{L}$-structure $\mathcal{N}$, $\mathcal{N} \vDash \theta$ if and only if there is an embedding $\mathfrak{a}_0 \to \mathcal{N}$.

Now, we as $X := \{f(n) : n < \omega\}$ is an infinite set and $\{f(n) : \mathrm{rk}(\mathfrak{a}_n) \geq e\}$ is infinite for every $e < \omega$, we may choose a non-principal ultrafilter $\mathscr{U}$ on ω such that $X \in \mathscr{U}$ and $\{k : \mathrm{rk}(\mathfrak{b}_k) \geq e\} \in \mathscr{U}$ for every $e < \omega$. As $\mathbf{C}$ is generically-categorical up to foundation rank and $\mathcal{M} \vDash \theta$, we know that $\prod_k \mathfrak{b}_k/\mathscr{U} \vDash \theta$, so

$$\big(X \cap \{k : \mathfrak{b}_k \vDash \theta\}\big) \setminus \{f(0)\} \in \mathscr{U}.$$

It follows that $\mathfrak{a}_n \vDash \theta$ for some $n > 0$, so $\mathfrak{a}_0 \leq^* \mathfrak{a}_n$. □

Lemma A.9. *[2⇒1]*

Proof. Assuming that $\mathbf{C}$ is well-quasi-ordered by $\leq^*$ up to foundation rank, suppose that $\mathbf{C}$ is *not* super-robust. Since $T_{\mathbf{C}}$ is model-complete and since $T_{\mathbf{C}} = T^{\infty}_{\mathbf{C}}$ by Lemma A.2, we find that there are an existential formula $\varphi(x_0, ..., x_{k-1}) \in \mathscr{L}$, structures $\mathfrak{a}_r, \mathfrak{b}_r \in \mathbf{C}(\mathcal{M})$, and $\bar{c}_r \in \|\mathfrak{a}\|^k$ $(r < \omega)$ such that for every $r < \omega$, $\mathrm{rk}(\mathfrak{a}_r) \geq r$, $\mathfrak{a}_r \leq \mathfrak{b}_r$, $\mathfrak{a}_r \vDash \neg\varphi(\bar{c}_r)$ and $\mathfrak{b}_r \vDash \varphi(\bar{c}_r)$. By the well-quasi-ordering assumption, we may assume that $\bar{c}_r = \bar{c}_0$ for all $r < \omega$, and by Corollary A.3, we may assume that $\bigcup_n \mathfrak{a}_n = \mathcal{M}$; it follows that $\mathcal{M} \vDash \neg\varphi(\bar{c}_0)$. On the other hand, since $\bigcup_n \mathfrak{a}_n = \mathcal{M}$, we also have $\bigcup_n \mathfrak{b}_n = \mathcal{M}$, so $\mathcal{M} \vDash \varphi(\bar{c}_0)$ – a contradiction. □

A.2. **Proof of the Corollary 3.10.**

Lemma A.10. *[1⇒5]*

Proof. Suppose $\mathbf{C}$ is a super-robust coherent class via $\nu : \mathscr{L} \to \omega$, and suppose that for every $e < \omega$, $\left\{\mathfrak{a}/\cong : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \leq e\right\}$ is finite. Towards a contradiction, suppose $\mathbf{C}$ *does not* admit an LS-function. In particular, suppose that $n < \omega$ is such that for every $k < \omega$, there are $\mathfrak{b} \in \mathbf{C}$ and $X \subseteq \|\mathfrak{b}\|$ such that $|X| \leq n$ but for every $\mathfrak{a} \leq \mathfrak{b}$, if $\mathfrak{a} \in \mathbf{C}$ and $X \subseteq \|\mathfrak{a}\|$, then $|\mathfrak{a}| > k$.

We may choose a family of pairs $(\mathfrak{b}_k, \overline{a}_k)$ $(k < \omega)$ such that for every k, $\overline{a}_k \in \|\mathfrak{b}_k\|^n$ and for every $\mathfrak{a} \leq \bar{b}_k$, if $\mathfrak{a} \in \mathbf{C}$ and $\mathfrak{a}_k \subseteq \|\mathfrak{a}\|$, then $|\mathfrak{a}| > k$. Without loss of generality, we may also assume that $|\mathfrak{b}_k| < |\mathfrak{b}_{k+1}|$ for all $k < \omega$.

We extend $\{\mathfrak{b}_k\}_k$ to a full set of representatives $X = X_0 \cup \{\mathfrak{b}_k\}_k$ of the isomorphism types in $\mathbf{C}$, and we take $\mathscr{U}$ to be a non-principal ultrafilter on ω such that $\{\mathfrak{b}_k\}_k \in \mathscr{U}$; set $\mathcal{N} = \Pi X/\mathscr{U}$. Further, for each $i < n$, let $f_i : X \to \bigcup X$ such that $f_i(\mathfrak{b}_k) = a_{k,i}$ for every $k < \omega$. Finally, for each $\mathfrak{a} \in \mathbf{C}$ with $|\mathfrak{a}| \geq n$ and an enumeration $<$ of $\|\mathfrak{a}\|$, let $\theta_{\mathfrak{a},<}(x_0, ..., x_{n-1}, \overline{y})$ be quantifier-free formula asserting the isomorphism type of $\mathfrak{a}$ relative to the given ordering of $\|\mathfrak{a}\|$. Then, for all $\mathfrak{a}$ and $<$, we have

$$\mathcal{N} \vDash \neg\exists\overline{y}\, \theta_{\mathfrak{a},<}(\overline{f}/\mathscr{U}, \overline{y}).$$

Since $\mathbf{C}$ is generically-categorical (as 1$\Leftrightarrow$3 is known to us), $\Pi X/\mathscr{U} \vDash T_{\mathbf{C}}$, so $\mathrm{tp}^{\mathcal{N}}(\overline{f}/\mathscr{U})$ is a non-isolated type over $T_{\mathbf{C}}$ – contradicting the hypothesis that $\mathbf{C}$ is coherent. This completes the proof of 1$\Rightarrow$5. $\square$

Lemma A.11. *[5$\Rightarrow$1]*

Proof. Once again, $\mathbf{C}$ is a robust semi-coherent class via $\nu_0 : \mathscr{L} \times \omega \to \omega$. We first show that under the hypotheses of corollary, an LS-function gives a somewhat stronger notion of robustness.

Claim. There is a function $\nu_1 : \mathscr{L} \times \omega \to \omega$ such that for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbf{C}$, $\varphi(x_0, ..., x_{n-1}) \in \mathscr{L}$, and $\overline{a} \in \|\mathfrak{a}\|^n$, if $\mathfrak{a} \leq \mathfrak{b} \leq \mathfrak{c}$ and $\mathrm{rk}(\mathfrak{b}) \geq \nu(\varphi, \mathrm{rk}(\mathfrak{a}))$, then $\mathfrak{b} \vDash \varphi(\overline{a}) \Leftrightarrow \mathfrak{c} \vDash \varphi(\overline{a})$.

Proof of claim (sketch). Under the assumption that $\left\{\mathfrak{a}/\cong : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \leq e\right\}$ is finite for every $e < \omega$, one may define $n_e = \max\left\{|\mathfrak{a}| : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \leq e\right\} < \omega$ for each $e < \omega$ and

$$r_n = \min\left\{r : (\forall \mathfrak{a} \in \mathbf{C})\left[\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq r \Rightarrow |\mathfrak{a}| \geq n\right]\right\} < \omega$$

for each n. Using these and the LS-function, one may define a function $\nu_1 : \mathscr{L} \times \omega \to \omega$ such that for all $\mathfrak{a}, \mathfrak{b} \in \mathbf{C}$ and $r < \omega$, if $\mathfrak{a} \leq \mathfrak{b}$, $\mathrm{rk}(\mathfrak{a}) \leq r$, and $\mathrm{rk}(\mathfrak{b}) \geq \nu_1(\varphi, r)$, then $\mathrm{rk}(\mathfrak{b}/\mathfrak{a}) \geq \nu_0(\varphi, r)$. Clearly, this suffices. $\square$

Using the LS-function and the fact that $\left\{\mathfrak{a}/\cong : \mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \leq e\right\}$ is finite for every $e < \omega$, we know that for every $0 < n < \omega$, three are quantifier-free formulas $\theta_0(x_0, ..., x_{n-1}, \overline{y}), ..., \theta_{k_n-1}(\overline{x}, \overline{y})$ of $\mathscr{L}$ such that for every $\varphi(\overline{x}) \in \mathscr{L}$, there is a set $S_\varphi \subseteq k_n$ such that

$$T_{\mathbf{C}} \vDash \underbrace{\forall\overline{x}\left(\varphi(\overline{x}) \leftrightarrow \bigvee_{i \in S_\varphi} \exists\overline{y}\, \theta_i(\overline{x}, \overline{y})\right)}_{\eta_\varphi}.$$

To see that this is true, one observes that if $(b_0, ..., b_{N-1})$ is an enumeration of some $\mathfrak{b} \in \mathbf{C}(\mathcal{M})$, where $\mathcal{M} \vDash T_{\mathbf{C}}$, then $\mathrm{qftp}^{\mathcal{M}}(\bar{b})$ isolates $\mathrm{tp}^{\mathcal{M}}(\bar{b})$; the LS-function, then, allows us to bound the number of variables needed in these θ's. Since $\mathbf{C}$ is robust, $T_{\mathbf{C}} = T^{\infty}_{\mathbf{C}}$, so there is a number $e_\varphi < \omega$ such that $\mathrm{rk}(\mathfrak{a}) \geq e_\varphi$ implies $\mathfrak{a} \vDash \eta_\varphi$. We therefore define $\nu(\varphi) = \max\{e_\varphi, e_{\neg\varphi}\}$.

Now, suppose $\mathfrak{a} \leq \mathfrak{b}$ are in $\mathbf{C}$, $\overline{a} \in \|\mathfrak{a}\|^n$, and $\varphi(x_0, ..., x_{n-1}) \in \mathscr{L}$; suppose $\mathrm{rk}(\mathfrak{a}) \geq \nu(\varphi)$. Then, we observe that

$$\mathrm{rk}(\mathfrak{a}) \geq \nu(\varphi) \geq e_\varphi \wedge \mathfrak{a} \vDash \varphi(\overline{a}) \Longrightarrow \mathfrak{b} \vDash \varphi(\overline{a})$$

and

$$\mathrm{rk}(\mathfrak{a}) \geq \nu(\varphi) \geq e_{\neg\varphi} \wedge \mathfrak{a} \vDash \neg\varphi(\overline{a}) \Longrightarrow \mathfrak{b} \vDash \neg\varphi(\overline{a}).$$

Thus, ν witnesses the fact that $\mathbf{C}$ is super-robust. $\square$

Appendix B. Anti-proficiency

[Connection to proof of 0,1-law for IFP/LFP.]

Definition B.1. Let $\varphi(x_0, ..., x_{n-1}, \overline{y}; R)$ be formula in the language $\mathscr{L}_R$, whose signature is $\mathrm{sig}(\mathscr{L}) \cup \{R^{(n)}\}$; also, assume that no variable in the tuple $\overline{y}$ occurs bound in φ.

$$\begin{aligned}\varphi^0(\overline{x}, \overline{y}) :=&\quad \neg(x_0 = x_0)\\ \varphi^{\alpha+1}(\overline{x}, \overline{y}) :=&\quad \varphi^\alpha(\overline{x}, \overline{y}) \vee \varphi\big(\overline{x}, \overline{y} : \varphi^\alpha(\overline{x}, \overline{y})\big)\\ \varphi^\lambda(\overline{x}, \overline{y}) :=&\quad \bigvee\nolimits_{\alpha<\lambda} \varphi^\alpha(\overline{x}, \overline{y}) \text{ if } \lambda \text{ is a limit}\end{aligned}$$

If $\mathcal{A}$ is an $\mathscr{L}$-structure, $\overline{b} \in A^{|\overline{y}|}$, and $\overline{a} \in A^n$,

$$\mathcal{A} \vDash \varphi^\infty(\overline{a}, \overline{b}) \Leftrightarrow \begin{cases} \mathcal{A} \vDash \varphi^{|A|}(\overline{a}, \overline{b}) & \text{if } \mathcal{A} \text{ is infinite} \\ \mathcal{A} \vDash \varphi^{(|A|+1)^{2n}}(\overline{a}, \overline{b}) & \text{if } \mathcal{A} \text{ is finite} \end{cases}$$

These definitions yield a chain of sets,

$$\varphi^0(\mathcal{A}, \overline{b}) \subseteq \cdots \subseteq \varphi^\alpha(\mathcal{A}, \overline{b}) \subseteq \cdots \subseteq \varphi^\infty(\mathcal{A}, \overline{b}) \subseteq A^n$$

We also define

$$\langle \mathcal{A}, \overline{b}, \varphi \rangle = \inf \left\{ \alpha : \varphi^\alpha(\mathcal{A}, \overline{b}) = \varphi^{\alpha+1}(\mathcal{A}, \overline{b}) \right\}.$$

Definition B.2. Let $\mathbf{C}$ be a class of finite $\mathscr{L}$-structures.

- We say that $\mathbf{C}$ is *proficient* if there is a formula $\varphi(x_0, ..., x_{n-1}, y_0, ..., y_{k-1}; R^{(n)})$ of the appropriate form such that for every $N < \omega$, there are $\mathfrak{a} \in \mathbf{C}$ and $\overline{b} \in \|\mathfrak{a}\|^k$ such that $\langle \mathfrak{a}, \overline{b}, \varphi \rangle \geq N$.
- We say that $\mathbf{C}$ is *anti-proficient* if for every formula $\varphi(x_0, ..., x_{n-1}, y_0, ..., y_{k-1}; R^{(n)})$, there is formula $\theta_\varphi(\overline{x}, \overline{y}) \in \mathscr{L}$ such that for every for all $\mathfrak{a} \in \mathbf{C}$ and $\overline{b} \in \|\mathfrak{a}\|^k$, $\varphi^\infty(\mathfrak{a}, \overline{b}) = \theta_\varphi(\mathfrak{a}, \overline{b})$.

Theorem B.3. *Every super-robust coherent class is anti-proficient.*

Proof. Let $\varphi(x_0, ..., x_{n-1}, y_0, ..., y_{k-1}; R^{(n)})$ be given. Using the $\aleph_0$-categoricity of $T_\mathbf{C}$ (from coherence), let $\psi_0(\overline{x}, \overline{y}), ..., \psi_{N-1}(\overline{x}, \overline{y})$ be principal formulas such that for every $p \in S_{n+k}(T_\mathbf{C})$, p is isolated by ψ_i for some $i < N$. Without loss of generality, we assume that the ψ_i's are pairwise inconsistent *modulo* $T_\mathbf{C}$. We define a function $f : \omega \times S_k(T_\mathbf{C}) \to \mathcal{P}(N)$ by setting

$$f(s, q) = \left\{ i < N : T_\mathbf{C} \vDash \psi_i(\overline{x}, \overline{b}) \to \varphi^s(\overline{x}, \overline{b}) \right\}$$

for some/any realization $\overline{b}$ of q. Since $\vDash \varphi^s(\overline{x}, \overline{y}) \to \varphi^{s+1}(\overline{x}, \overline{y})$ for all $s < \omega$, for each $q \in S_k(T_\mathbf{C})$, there is a non-empty set $Y_q \subseteq N$ such that $f(s, q) = Y_q$ for all but finitely many $s < \omega$. Then, we set

$$\theta_\varphi(\overline{x}, \overline{y}) = \bigvee_{q \in S_k(T_\mathbf{C})} \bigvee_{i \in Y_q} \psi_i(\overline{x}, \overline{y})$$

and we find that for any model $\mathcal{M} \vDash T_\mathbf{C}$ and any $\overline{b} \in M^k$, $\varphi^\infty(\mathcal{M}, \overline{b}) = \theta_\varphi(\mathcal{M}, \overline{b})$. By super-robustness, there is a number $r < \omega$ such that for all $\mathfrak{a} \in \mathbf{C}$ and all $\overline{b} \in \|\mathfrak{a}\|^k$, if $\mathrm{rk}(\mathfrak{a}) \geq r$, then $\varphi^\infty(\mathfrak{a}, \overline{b}) = \theta_\varphi(\mathfrak{a}, \overline{b})$. Since the $\{\mathfrak{a} \in \mathbf{C} : \mathrm{rk}(\mathfrak{a}) < r\}$ is finite up to isomorphism, this completes the proof of anti-proficiency. $\square$