Some model-theoretic remarks on structural Ramsey theory

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Abstract

We present one novel result and two novel proofs of previously known results in structural Ramsey theory. Regarding the former, we give a new characterization of the Ramsey property for a Fraïssé class in terms colorings of induced substructures of its generic model (or Fraïssé limit). This result is obtained as a corollary of a more general theorem in which we show that under relatively mild hypothesis, every expansion $\mathcal{N}$ of a given $\aleph_0$-categorical structure $\mathcal{M}$ induces at least one generic expansion of $\mathcal{M}$ (roughly in the sense of [4]) "constrained by $\mathcal{N}$."

As for new proofs of old facts, we give a new proof the famous theorem of [5] asserting that a Fraïssé class $\mathbf{K}$ has the Ramsey property if and only if the automorphism group of its generic model is extremely amenable. Here, we use very explicitly model-theoretic techniques to show that that a Fraïssé class $\mathbf{K}$ has the Ramsey property if and only if the automorphism group of its generic model is extremely amenable relative to Stone spaces. Finally, we give a novel proof of the fact that the generic model of a Ramsey class always carries a 0-definable linear ordering; this new demonstration makes essential appeals to the Ramsey property's role in constructing generalized indiscernible sequences and generalized Ehrenfeucht-Mostowski models.

Introduction

In this paper, we make two contributions. The first of these extends the list of alternative characterizations of the Ramsey property for a class $\mathbf{K}$ of finite structures by qualities of its generic structure $\mathcal{M}$ (Fraïssé limit, when $\mathbf{K}$ is a Fraïssé class). We give two characterizations of the Ramsey property in terms of colorings of copies of finite induced substructures of $\mathcal{M}$ – the amalgamated Ramsey property and the generic infinitary Ramsey property. In the latter, the more interesting of these, we consider whether or not, given any *generic coloring* $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$, there is an elementary embedding $f : \mathcal{M} \to \mathcal{M}$ such that ξ is constant on $\mathrm{Emb}(A, f\mathcal{M})$, where $\mathrm{Emb}(A, \mathcal{M})$ denotes the set of all embeddings $A \to \mathcal{M}$.

To effect this characterization, we prove a more general theorem about generic expansions of certain $\aleph_0$-categorical structures, showing that if $\mathcal{M}$ is obtained as the generic structure of a "strongly coherent class" and $\mathcal{N}$ is an expansion of $\mathcal{M}$ (by finitely many new relations), then there is a generic expansion $\mathcal{M}'$ of $\mathcal{M}$ (roughly in the sense of [4]) that is constrained by $\mathcal{N}$. Here, "constrained by $\mathcal{N}$" means essentially that every finite induced substructure occurs as a finite induced substructure of $\mathcal{N}$ up to isomorphism (although not all substructures of $\mathcal{N}$ must occur). This technology of constrained generic expansions does not depend upon the Ramsey property, and we believe it is of independent interest.

Our second contribution amounts to re-proving some previously known facts by quite different means. First, we give a new proof of the fact that the generic model of a Ramsey class (when it is a true Fraïssé class) is definably linearly ordered by a quantifier-free formula. This is known from [5], but our proof eschews continuous group actions and logic topologies in favor of constructions of generalized indiscernible sequences and generalized Ehrenfeucht-Mostowski models. Within our proof of linear ordering, we also give a somewhat novel proof using our constrained generic expansions technology of a theorem of [9], in which the Ramsey property is characterized in relation to constructing generalized indiscernibles. As a small benefit, we find that "generic" is the weak cousin of "indiscernible" that can be recovered for any Fraïssé class, not just Ramsey classes. In the second batch of new proofs, we give a very simple proof of the fact that the Ramsey property of $\mathbf{K}$ is sufficient for extreme amenability of $Aut(\mathcal{M})$ relative Stone spaces (EA/Stone). From this starting point, we give a new proof of the fact that extreme amenability is equivalent to EA/Stone, proved first in [1], and our methods are much more clearly model-theoretic in nature than others of which the author is aware.

1 Amalgamation classes and the Ramsey property

We introduce almost all of the notation and necessary background material for our investigation of manufactured dimension functions. Subsection 1.1 accounts for just some basic notation that might not be completely standardized in the literature. In Subsection 1.2, we review definitions and basic results around amalgamation classes of finite structures and their generic models, and we present the definition of the Ramsey property in its standard "finitary" form. and begin the discussion of "infinitary" versions. We discuss the similarities and differences between our notion of generic expansion and that used in [4], and we discuss a naïve approach to defining an infinitary Ramsey property that does not work but in its failure suggests appropriate modifications that could lead to new characterizations. Thereafter, in Subsection 1.3, we define our two infinitary Ramsey properties – the amalgamated Ramsey property and the generic infinitary Ramsey property – and prove some basic facts about them.

1.1 Notation and conventions

Unless explicitly stated otherwise, the signature $\mathrm{sig}(\mathscr{L})$ of any language $\mathscr{L}$ in question consists of countably many relation symbols, finitely many constant symbols, and no function symbols. Infinite structures are denoted by calligraphic upper-case letters like $\mathcal{A}$ and $\mathcal{M}$ with universes A and M, respectively, and in general, our notation for such structures is more or less standard (see [7]). For finite structures, we use simple upper-case letters like A, B, C, and we identify finite structures with their universes. For a subset A of M, where $\mathcal{M}$ is an infinite structure, $\mathcal{M}[A]$ denotes the induced substructure of $\mathcal{M}$ with universe A (together with interpretations of constant symbols, if any), but we often write A instead of $\mathcal{M}[A]$ if no confusion is likely to arise.

A class of finite structures is usually denoted by $\mathbf{K}$ or some other bold upper-case letter, and these are always assumed to be closed under isomorphisms. We also assume that for every n, the set $\{A \in \mathbf{K}_\forall : |A| = n\} /\!\cong$ is finite, where $\mathbf{K}_\forall = \{A : A \leq B \in \mathbf{K}\}$.

If $\mathcal{M}$ is a structure and g is a permutation of M, then $g\mathcal{M}$ is the structure with universe $\mathcal{M}$ and interpretations $R^{g\mathcal{M}} = \left\{g\overline{a} : \overline{a} \in R^{\mathcal{M}}\right\}$, $c^{g\mathcal{M}} = g(c^{\mathcal{M}})$. If A is a finite structure with $A \subset M$ and g is a permutation of M, gA is defined similarly. If $\mathcal{M}$ is an $\aleph_0$-categorical structure, then we define

$$\mathrm{acl}[\mathcal{M}] = \{\mathrm{acl}(X) : X \subset_{\mathrm{fin}} M\}\,.$$

When $\mathcal{M}$ is the generic model of an amalgamation class $\mathbf{K}$ (see below), then $\mathbf{K}(\mathcal{M})$ is the set of induced substructures of $\mathcal{M}$ that are in $\mathbf{K}$.

1.2 Amalgamation classes

In this subsection, we defined several amalgamation-type properties that a class of finite structures might have. Conjoining several of these properties in different combinations yields the notions of Fraïssé classes and coherent classes which are the setting for structural Ramsey theory as discussed in this paper. Almost all of these properties are gathered together in Definition 1.1, and all of them have been remarked upon by other authors. A key fact for all of these conjunctions is the existence of generic models (sometimes called Fraïssé limits in certain cases)

Definition 1.1. Let $\mathbf{K}$ be a class of finite $\mathscr{L}$-structures. We list a number of properties that $\mathbf{K}$ might have.

1. (Joint-embedding property (JEP))

   For any $A, B \in \mathbf{K}$, there are $C \in \mathbf{K}$ and embeddings $f_A : A \to C$ and $f_B : B \to C$.

2. (Amalgamation property (AP))

   For any $A, B_1, B_2 \in \mathbf{K}$ and embeddings $f_i : A \to B_i$ $(i = 1, 2)$, there are $C \in \mathbf{K}$ and embeddings $f_i' : B_i \to C$ such that $f_1' f_1 = f_2' f_2$.

3. (Disjoint joint-embedding property (disjoint-JEP))

For any $A, B \in \mathbf{K}$, there are $C \in \mathbf{K}$ and embeddings $f_A : A \to C$ and $f_B : B \to C$ such that $f_A A \cap f_B B = \emptyset$.

4. (Disjoint amalgamation property (disjoint-AP))

   For any $A, B_1, B_2 \in \mathbf{K}$ and embeddings $f_i : A \to B_i$ $(i = 1, 2)$, there are $C \in \mathbf{K}$ and embeddings $f_i' : B_i \to C$ such that $f_1' f_1 = f_2' f_2$ and $f_1' B_1 \cap f_2' B_2 = f_1' f_1 A = f_2' f_2 A$.

5. (Heredity property (HP))

   For every $B \in \mathbf{K}$, every induced substructure $A \leq B$ is in $\mathbf{K}$.

6. (Weak Löwenheim-Skolem property (WLSP))

   There is a function $\lambda : \mathbb{N} \to \mathbb{N}$ such that for any $A \in \mathbf{K}$ and any $X \subseteq A$, there are $A', B \in \mathbf{K}$ such that $A, A' \leq B$, $X \subseteq A'$, and $|A'| \leq \lambda(|X|)$.

We say that $\mathbf{K}$ is an *amalgamation class* if it has JEP and AP. As always, $\mathbf{K}$ is a *Fraïssé class* if it has JEP, AP, and HP. A class $\mathbf{K}$ is a *coherent class* if it has JEP, AP, and WLSP.

Theorem 1.2 (cf. Theorem 7.1.2 of [3]). *Let* $\mathbf{K}$ *be a class of finite structures. If* $\mathbf{K}$ *has both AP and JEP, then there is a countably infinite* generic model $\mathcal{M}$ *with the following three properties:*

- *(*$\mathbf{K}$*-universality) For every* $A \in \mathbf{K}$*, there is an embedding* $A \to \mathcal{M}$*.*
- *(*$\mathbf{K}$*-homogeneity) For any* $A, B \in \mathbf{K}$ *and any embedding* $f_0 : A \to \mathcal{M}$*, there is an embedding* $f : B \to \mathcal{M}$ *such that* $f_0 \subseteq f$*.*
- *(*$\mathbf{K}$*-closedness) For every* $X \subset_{fin} M$*, there are* $A \in \mathbf{K}$ *and an embedding* $f : A \to \mathcal{M}$ *such that* $X \subseteq fA$*.*

Any countable structure with these three properties is isomorphic to $\mathcal{M}$*; because of this uniqueness, we call* $\mathcal{M}$ the *generic model, and we see that the generic theory* $T_{\mathbf{K}} = Th(\mathcal{M})$ *is well-defined in terms of* $\mathbf{K}$*. Furthermore:*

- *[6]:* $T_{\mathbf{K}}$ *is* $\aleph_0$*-categorical and model-complete if and only if* $\mathbf{K}$ *has WLSP.*
- *[3]:* $T_{\mathbf{K}}$ *is* $\aleph_0$*-categorical and eliminates quantifiers if and only if* $\mathbf{K}_\forall$ *has AP and JEP (i.e.* $\mathbf{K}_\forall$ *is a Fraïssé class).*
- $\mathbf{K}$ *has disjoint-JEP if and only if* $\mathrm{acl}(\emptyset) = \emptyset$ *in* $\mathcal{M}$*.*
- $\mathbf{K}$ *has disjoint-AP if and only if for every embedding* $f : A \to \mathcal{M}$*, where* $A \in \mathbf{K}$*,* fA *is algebraically closed.*

For our purposes in this paper, coherent classes may be a little too general, so we will focus on the narrower concept of strongly coherent class. Directly from the definition, it is not difficult to see that a Fraïssé class with disjoint-JEP and disjoint-AP is necessarily a strongly coherent class.

Definition 1.3. Let $\mathbf{K}$ be a class of finite structures. We say that $\mathbf{K}$ is a *strongly coherent class* if it has disjoint-AP, WLSP, and if $\mathcal{M}$ is the generic model of $\mathbf{K}$, then $\mathrm{acl}[\mathcal{M}] \subseteq \mathbf{K}$. (The last condition is equivalent to requiring that $\mathbf{K}$ is the isomorphism-closure of $\mathrm{acl}[\mathcal{M}]$.) A strongly coherent class that is actually a Fraïssé class could be called a *strong Fraïssé class*.

We now define formally what we mean by a generic expansion of a given structure and by a constrained generic expansion. After discussing the generic infinitary Ramsey property (GRP) in Subsection 1.3 below, we will use this concept in a more general theorem (proved in Section 1.16) that ensures that GRP does in fact characterize the Ramsey property.

Definition 1.4. Let $\mathcal{M}$ be the generic model of a strongly coherent class $\mathbf{K}$ of $\mathscr{L}$-structures. Let $\mathscr{L}' \supseteq \mathscr{L}$, and let $\mathcal{N}, \mathcal{N}_0$ be $\mathscr{L}'$-expansions of $\mathcal{M}$. Then we define

$$\mathsf{Age}(\mathcal{N}/\mathcal{M}) = \{(g\mathcal{N})[A] : A \in \mathbf{K}(\mathcal{M}),\, g \in Aut(\mathcal{M})\}\,.$$

Using $\mathsf{Age}(\mathcal{N}/\mathcal{M})$, we make several more definitions:

- We say that $\mathcal{N}_0$ *is a generic expansion of* $\mathcal{M}$ if (after closing under isomorphisms) $\mathsf{Age}(\mathcal{N}_0/\mathcal{M})$ is a coherent class with disjoint-AP.

  In more painful detail: $\mathcal{N}_0$ is a generic expansion of $\mathcal{M}$ if for any $A, A' \in \mathbf{K}(\mathcal{M})$, any $g \in Aut(\mathcal{M})$, and any $gA \subseteq X \subset_{\text{fin}} M$, if $A \leq A'$ (as $\mathscr{L}$-structures) and $g{\restriction}A$ is an $\mathscr{L}'$-isomorphism of $\mathcal{N}_0[A]$ onto $\mathcal{N}_0[gA]$, then there is an automorphism $h \in Aut(\mathcal{M})$ such that $g{\restriction}A \subseteq h$, $hA' \cap X = gA$, and which is an $\mathscr{L}'$-isomorphism of $\mathcal{N}_0[A']$ onto $\mathcal{N}_0[hA']$.

- We say that $\mathcal{N}_0$ *is a generic expansion of* $\mathcal{M}$ *constrained by* $\mathcal{N}$ if (i) $\mathcal{N}_0$ is a generic expansion of $\mathcal{M}$ and (ii) $\mathsf{Age}(\mathcal{N}_0/\mathcal{M}) \subseteq \mathsf{Age}(\mathcal{N}/\mathcal{M})$.

To conclude this subsection, we formalize the Ramsey property as it pertains to strongly coherent classes. This is the classical, finitary formulation of this property that we will attempt the characterize in terms of certain properties of the generic model.

Definition 1.5. Let $\mathbf{K}$ be a strongly coherent class.

- For $A \in \mathbf{K}$, we say that $\mathbf{K}$ has the A-*Ramsey property* if for every $B \in \mathbf{K}$ and every $k \in \mathbb{N}^+$, there is a $C \in \mathbf{K}$ such that $C \to (B)^A_k$, meaning that for every coloring $\xi : \mathrm{Emb}(A, C) \to [k]$, there is an embedding $u : B \to C$ such that ξ is constant in $\mathrm{Emb}(A, uB)$. (Here, $[k] = \{0, 1, ..., k-1\}$ for every $k \in \mathbb{N}^+$.)

- We say that $\mathbf{K}$ has the *Ramsey property* (and that $\mathbf{K}$ is a Ramsey class) if it has the A-Ramsey property for every $A \in \mathbf{K}$.

Fact 1.6 (Theorem from [8]). Let $\mathbf{K}$ be a strongly coherent class. If $\mathbf{K}$ has the Ramsey property, then it is finitely-rigid in the sense that for every $A \in \mathbf{K}$, $Aut(A) = \{id_A\}$.

We note that the quantifier structure of the Ramsey property is $\forall\forall\forall\exists\forall\exists\forall\forall$, which while certainly insurmountable, is not especially easy to deal with. One advantage one might hope for in formulating infinitary Ramsey properties might lie just decreasing the complexity of the quantifier structure, even cosmetically.

1.2.1 The Baire-categorical explanation of the word "generic" and the connection to "generic expansions" in the sense of [4].

In this paper, we are beholden to [4] for our notion of generic expansions, but our definition is not identical to the one used there. In this subsection, we will discuss how the differ. For clarity, we will say that $\mathcal{N}$ *is an I-generic expansion of* $\mathcal{M}$ when we mean to use the definition from [4]. The notion of "generic expansion" used in the present paper seems to be more restrictive than that of "I-generic expansion" (certainly so if $\mathcal{M}$ is the generic model of a strong Fraïssé class). In fact, our notion of "generic expansion" is somewhat closer to the notions of "reasonable ordering" of [5] and "respectful expansions" of [2].

Let $\mathcal{M}$ be an $\aleph_0$-categorical $\mathscr{L}_0$-structure, and let $\mathscr{L}_1$ be an extension of $\mathscr{L}_0$ by finitely many new relation symbols and no new constant symbols. For this discussion, it is convenient that $\mathscr{L}$ has no constant symbols. We recover a topological space $\mathbb{X}$ and a number of subspaces, each equipped with an $Aut(\mathcal{M})$-action by homeomorphisms, as follows:

- Let $\mathbb{X}$ denote the set of all $\mathscr{L}_1$-expansions of $\mathcal{M}$, and let $J = \{\mathcal{N}[A] : A \subset_{\text{fin}} M, \mathcal{N} \in \mathbb{X}\}$, a set finite $\mathscr{L}_1$-structures expanding finite induced substructures of $\mathcal{M}$.

  For each $B \in J$, $[B]$ denotes the set of all $\mathcal{N} \in \mathbb{X}$ such that $B \leq \mathcal{N}$. Then $\{[B] : B \in J\}$ forms a sub-base for a Cantor topology on $\mathbb{X}$. As a Cantor space, $\mathbb{X}$ is compact Hausdorff, so it is a Baire space (countable intersections of co-meagre sets are dense).

  There is also a natural action $Aut(\mathcal{M}) \curvearrowright \mathbb{X}$ by homeomorphisms as described in Subsection 1.1.

- Now, let $\mathcal{N} \in \mathbb{X}$. Then we define

  $$J^{\mathcal{N}} = \{(g\mathcal{N})[A] : g \in Aut(\mathcal{M}),\ A \subset_{\text{fin}} M\}$$

  a closed G_δ set

  $$V(\mathcal{N}) = \bigcap_{A \subset_{\text{fin}} M} \bigcup_{g \in Aut(\mathcal{M})} \Big[(g\mathcal{N})[A]\Big] = \left\{\mathcal{N}' \in \mathbb{X} : (\forall A \subset_{\text{fin}} M)\, \mathcal{N}'[A] \in J^{\mathcal{N}}\right\}.$$

  Then $V(\mathcal{N})$ consists of those $\mathcal{N}'$ that, up to $Aut(\mathcal{M})$, do not have any finite induced substructures other than those that occur as induced substructures of $\mathcal{N}$.

Remark 1.7. In the sense of [4], $\mathcal{N} \in \mathbb{X}$ is an I-generic expansion of $\mathcal{M}$ if and only if the orbit $\mathcal{N}^{Aut(\mathcal{M})}$ is a co-meagre subset of $V(\mathcal{N})$. One of the main theorems of that paper is the following:

> *Theorem 1.2 of [4].* Let $\mathcal{N} \in \mathbb{X}$. Then, $\mathcal{N}$ is an I-generic expansion of $\mathcal{M}$ if and only if $J^{\mathcal{N}}$ has the following properties:
>
> 1. Joint-embedding property: For any $A, B \in J^{\mathcal{N}}$, there are $g \in Aut(\mathcal{M})$ and $C \in J^{\mathcal{N}}$ such that $A \leq C$ and $gB \leq C$.

2. Almost-amalgamation property: For every $A \in J^{\mathcal{N}}$, there is an $A^* \in J^{\mathcal{N}}$ such that $A \leq A^*$ and for all $B_1, B_2 \in J^{\mathcal{N}}$, if $A^* \leq B_1$ and $A^* \leq B_2$, then there are $g \in Aut(\mathcal{M}/A)$ and $C \in J^{\mathcal{N}}$ such that $A \leq B_1 \leq C$ and $gB_2 \leq C$.

If $\mathcal{M}$ is the generic model of a strongly coherent class $\mathbf{K}$ and $\mathcal{N} \in \mathbb{X}$, then to move from I-generic to generic expansions in our sense, one makes the following replacements:

- Use $\mathsf{Age}(\mathcal{N}/\mathcal{M})$ instead of $J^{\mathcal{N}}$ everywhere in the discussion.
- Replace the almost-amalgamation property with the following version of disjoint-AP (alt-disjoint-AP):

  For every $A \in \mathsf{Age}(\mathcal{N}/\mathcal{M})$, for all $B_1, B_2 \in \mathsf{Age}(\mathcal{N}/\mathcal{M})$, if $A \leq B_1$ and $A \leq B_2$, then there are $g \in Aut(\mathcal{M}/A)$ and $C \in \mathsf{Age}(\mathcal{N}/\mathcal{M})$ such that $B_1 \leq C$, $gB_2 \leq C$, and $B_1 \cap gB_2 = A$.

Then by definition, $\mathcal{N}$ is a generic expansion of $\mathcal{M}$ if and only if $\mathsf{Age}(\mathcal{N}/\mathcal{M})$ has both the joint-embedding property and alt-disjoint-AP as just described. It's easy to see that alt-disjoint-AP implies the almost-amalgamation property (just take $A^* = \mathrm{acl}(A)$), so if $\mathcal{N}$ is a generic expansion of $\mathcal{M}$, then $\mathcal{N}^{Aut(\mathcal{M})}$ is a co-meagre subset of $V(\mathcal{N})$. But in general, there are $\mathcal{N} \in \mathbb{X}$ for which $\mathcal{N}^{Aut(\mathcal{M})}$ is a co-meagre subset of $V(\mathcal{N})$ yet are not generic expansions of $\mathcal{M}$ in our sense.

1.2.2 The naïve approach to an "infinitary" Ramsey property does not work

The most obvious attempt to a formulate a infinitary Ramsey property for a strongly coherent class $\mathbf{K}$ with generic model $\mathcal{M}$ might be the following: "For every $A \in \mathbf{K}$, for every coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ (where $k \in \mathbb{N}^+$), there is an embedding $f : \mathcal{M} \to \mathcal{M}$ such that ξ is constant on $\mathrm{Emb}(A, f\mathcal{M})$." This naïve approach does not work even barely. Not only does this statement fail to characterize the Ramsey property, it is essentially never true at all, as we will see below.

The proof of the next proposition is an adaptation of a proof of the fact that $\mathbb{Q} \not\to (\mathbb{Q})^2_2$ (which appears to be folklore). The proof of the special case was communicated to us by L. Scow.

Proposition 1.8. *Let $\mathbf{K}$ be a strong Fraïssé class with generic model $\mathcal{M}$ such that $|S_1(T_{\mathbf{K}})| = 1$. Suppose there is a binary relation symbol $<$ in the language $\mathscr{L}$ of $\mathbf{K}$ that is interpreted in $\mathcal{M}$ as a linear ordering of M. (Hence, $(M, <^{\mathcal{M}}) \cong \mathbb{Q} \cap (0, 1)$.) Then, there are a structure $A \in \mathbf{K}$ and a coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to \{0, 1\}$ such that for every embedding $f : \mathcal{M} \to \mathcal{M}$, there are $u, u' \in \mathrm{Emb}(A, \mathcal{M})$ such that $\xi(fu) \neq \xi(fu')$.*

Proof. Let $\ell : \mathbb{Q} \cap (0, 1) \to (M, <^{\mathcal{M}})$ be an order-isomorphism. For each integer $n \geq 2$, we define

$$F_n = \left\{ \frac{k}{n} : k \in \{1, 2, ..., n-1\}, \gcd(k, n) = 1 \right\}$$

Then, $\mathbb{Q} \cap (0,1) = \dot{\bigcup}_{n \geq 2} F_n$, and we may define $\mathrm{rk} : \mathbb{Q} \cap (0,1) \to \omega$ by, $\mathrm{rk}(q) = n \Leftrightarrow q \in F_n$. Finally, define an auxiliary ordering $<_F$ on $\mathbb{Q} \cap (0,1)$ by

$$q <_F r \Leftrightarrow \bigvee \begin{cases} \mathrm{rk}(q) < \mathrm{rk}(r) \\ (\mathrm{rk}(q) = \mathrm{rk}(r) \wedge q < r). \end{cases}$$

Let $A \in \mathbf{K}(\mathcal{M})$ with $|A| \geq 3$. Let a_0, a_1 be the maximum and minimum elements of A, respectively, and let $c \in A \setminus \{a_0, a_1\}$. Now, we define a coloring $\xi : \mathrm{Hom}(A, \mathcal{M}) \to 2$ by

$$\xi(e) = \begin{cases} 0 & \text{if } \ell^{-1}\big(e(c)\big) <_F \ell^{-1}\big(e(a_1)\big) \\ 1 & \text{if } \ell^{-1}\big(e(a_1)\big) <_F \ell^{-1}\big(e(c)\big) \end{cases}$$

We will show that there is *no* embedding $f : \mathcal{M} \to \mathcal{M}$ such that ξ is constant on $\mathrm{Hom}(\mathfrak{a}, f\mathcal{M})$. For a contradiction, suppose f were such an embedding. Assume $f(c) = u(r)$, $f(a_0) = \ell(q_0)$ and $f(a_1) = \ell(q_1)$. There are two cases to account for:

- Case: $r <_F q_1$ – i.e. $\xi(f \restriction A) = 0$.

  Certainly, $q_1 \in F_n$ for some $n \geq 2$. There are infinitely many embeddings $e_i : A \to f\mathcal{M}$ $(i = 0, 1, ...)$ such that

  $$e_i(a_0) = f(a_0) <^{\mathcal{M}} e_i(c) <^{\mathcal{M}} e_{i+1}(c) <^{\mathcal{M}} f(a_1) = e_i(a_1)$$

  for all i. Since F_n is a finite set, there are i's such that $\mathrm{rk}\left(\ell^{-1}(e_i(c))\right) > n$ – in which case, $\xi(e_i) = 1$.

- Case: $q_1 <_F r$ – i.e. $\xi(f \restriction A) = 1$.

  In this case, $r \in F_n$ for some $n \geq 2$, and we can choose infinitely many embeddings $e_i : A \to f\mathcal{M}$ $(i = 0, 1, ...)$ such that $e_i(a_0) = f(a_0)$, $e_i(c) = f(c)$ and

  $$f(c) <^{\mathcal{M}} e_i(a_1) <^{\mathcal{M}} e_{i+1}(a_1)$$

  for all i. Again, since F_n is finite, there are $i < \omega$ such that $\mathrm{rk}\left(u^{-1}(e_i(a_1))\right) > n$ – in which case, $\xi(e_i) = 0$.

□

There are two ways to modify the naïve proposal that might remedy its failure. One modification would be to relax the requirement for an embedding $f : \mathcal{M} \to \mathcal{M}$ to just a family of embeddings $f_B : B \to \mathcal{M}$ for B ranging over $\mathbf{K}(\mathcal{M})$; a formalization of this idea will appear below as the "amalgamated Ramsey property," and it *is* indeed a characterization of the Ramsey property. An alternative modification is to restrict the kind of colorings that must be addressed (while retaining the requirement for a true embedding $\mathcal{M} \to \mathcal{M}$); a formalization of this idea will become the generic infinitary Ramsey property below, which will also characterize Ramsey classes.

1.3 Two infinitary Ramsey properties that work.

Our first "infinitary" Ramsey property arises from following the first path to remedying the failure of the naïve approach. Here, given a coloring, we ask not for true embeddings $\mathcal{M} \to \mathcal{M}$ but for certain kinds of systems of embeddings of finite substructures of $\mathcal{M}$ into $\mathcal{M}$.

Definition 1.9 (Amalgamated Ramsey property)**.** Let $\mathbf{K}$ be a strongly coherent class with generic model $\mathcal{M}$.

- A $\mathbf{K}$-embedding system (on $\mathcal{M}$) is a family $f = (f_B : B \in \mathbf{K}(\mathcal{M}))$ such that for each $B \in \mathbf{K}(\mathcal{M})$, f_B is an embedding $B \to \mathcal{M}$.
- Let $A \in \mathbf{K}$. We say that $\mathcal{M}$ has the *amalgamated A-Ramsey property* (A-ARP) if for every $k \in \mathbb{N}^+$ and every coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$, there is a $\mathbf{K}$-embedding system $f = (f_B)_B$ on $\mathcal{M}$ such that for every $B \in \mathbf{K}(\mathcal{M})$, ξ is constant on $\mathrm{Emb}(A, f_B[B])$.
- We say that $\mathcal{M}$ has the *amalgamated Ramsey property* (ARP) if it has the amalgamated A-Ramsey property for every $A \in \mathbf{K}$.

(The quantifier structure of the amalgamated Ramsey property is $\forall\forall\forall\exists\forall\forall$, which does appear to be a mild improvement on that of the finitary Ramsey property.)

Before moving to our second "infinitary" Ramsey property, we pause to prove that that the amalgamated Ramsey property truly provides a characterization of Ramsey classes.

Proposition 1.10. *Let $\mathbf{K}$ be a strongly coherent class with generic model $\mathcal{M}$. The following are equivalent:*

1. *$\mathbf{K}$ has the Ramsey property.*
2. *$\mathcal{M}$ has the amalgamated Ramsey property relative to $\mathbf{K}$.*

Proof of 1⇒2. It suffices to show that for each $A \in \mathbf{K}$, if $\mathbf{K}$ has the A-Ramsey property, then $\mathcal{M}$ has the amalgamated A-Ramsey property. Assuming the former, let $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ be some coloring. We must define the appropriate $\mathbf{K}$-embedding system $(f_B)_{B \in \mathbf{K}(\mathcal{M})}$, so let $B \in \mathbf{K}(\mathcal{M})$ be given. Applying the A-Ramsey property, we select $C \in \mathbf{K}$ such that $C \to (B)^A_k$, and we take v to be any embedding $C \to \mathcal{M}$. Then, there is an embedding $u : B \to C$ such that ξ is constant on $\mathrm{Emb}(A, uvB)$, and we set $f_B = uv$. □

Proof of 2⇒1. Taking $A \in \mathbf{K}$ as given, we assume that $\mathcal{M}$ has the amalgamated A-Ramsey property. Towards a contradiction, suppose that $\mathbf{K}$ does not have the A-Ramsey property. We are given, then, $B \in \mathbf{K}$ and $k \in \mathbb{N}^+$ such that $C \not\to (B)^A_k$ for all $C \in \mathbf{K}$.

Let $C_0, ..., C_n, ...$ be members of $\mathbf{K}(\mathcal{M})$ such that $C_n \leq C_{n+1}$ for all n and $\bigcup_n C_n = M$. For each n, since $C_n \not\to (B)^A_k$, there is a coloring $\xi_n : \mathrm{Emb}(A, C_n) \to [k]$ for which there is no embedding $v : B \to C_n$ such that ξ_n is constant on $\mathrm{Emb}(A, vB)$.

Let $\mathscr{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{N} = \prod_n \mathcal{M}/\mathscr{U}$. We define a coloring $\xi^* : \mathrm{Emb}(A, \mathcal{N}) \to [k]$ as follows. Let $u \in \mathrm{Emb}(A, \mathcal{N})$; we may view u as a family of maps $u_\bullet(a) : \mathbb{N} \times A \to M : (n, a) \mapsto u_n(a)$ so that $u(a) = (u_n(a))_n/\mathscr{U}$ for each $a \in A$. Then $X = \{n : u_n{=}(a \mapsto u_n(a)) \in \mathrm{Emb}(A, C_n)\}$ is in $\mathscr{U}$. For each $i \in [k]$, let $X_i = \{n \in X : \xi_n(u_n) = i\}$, so that X is the disjoint union of $X_1, ..., X_k$. Since $\mathscr{U}$ is an ultrafilter, there is a single $i \in [k]$ such that $X_i \in \mathscr{U}$, and we set $\xi^*(u) = i$. By construction, there is no embedding $v : B \to \mathcal{N}$ such that ξ^* is constant on $\mathrm{Emb}(A, vB)$.

For each $m \in M$, we define $w_m \in \prod_n C_n \subseteq \prod_n M$ by setting $w_n(m) = m$ for all n such that $m \in C_n$ (which is all but finitely many n). It is not hard to verify that $q : \mathcal{M} \to \mathcal{N} : m \mapsto w_\bullet(m)/\mathscr{U}$ is an embedding of $\mathscr{L}$-structures. Now, we can define a coloring $\zeta : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ by $\zeta(u) = \xi^*(qu)$. Since $\mathcal{M}$ has the amalgamated A-Ramsey property, there is an embedding $f : B \to \mathcal{M}$ such that ζ is constant $\mathrm{Emb}(A, fB)$. But it follows that ξ^* is constant on $\mathrm{Emb}(A, qfB)$ – a contradiction. Thus, $\mathbf{K}$ must have the A-Ramsey property, as claimed. $\square$

Thus, the amalgamated Ramsey property does actually work as advertised, but it is not especially satisfying as an alternative to the finitary Ramsey property. Certainly, the equivalence is not particularly surprising, and we would still like to see a version of the Ramsey property that treats the generic model on its own terms – that is, a version that does require embeddings $\mathcal{M} \to \mathcal{M}$ even if we must restrict the kinds of colorings allowed to enter into the discussions.

We will see that restricting attention to generic colorings allows us to construct true embeddings $\mathcal{M} \to \mathcal{M}$ instead of (in fact, out of) $\mathbf{K}$-embedding systems. We formulate now exactly what a generic coloring is.

Definition 1.11 (Generic colorings)**.** Let $\mathbf{K}$ be a finitely-rigid strongly coherent class of finite $\mathscr{L}$-structures with generic model $\mathcal{M}$. Let $A \in \mathbf{K}$, and let $k \in \mathbb{N}^+$. Assume $A = \{a_1, ..., a_r\}$

- Let $\mathscr{L}^k_A$ be the language extending $\mathscr{L}$ by k new relation symbols $R_1, ..., R_k$, each of arity $r = |A|$.
- Given a coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$, we make an $\mathscr{L}^k_A$-expansion $\mathcal{M}_\xi$ of $\mathcal{M}$ by setting
$$R_i^{\mathcal{M}_\xi} = \left\{ \left(u(a_{\sigma(1)}), ..., u(a_{\sigma(r)})\right) : \sigma \in Sym(r),\, u \in \mathrm{Emb}(A, \mathcal{M}),\, \xi(u) = i \right\}$$
for each $i \in [k]$.
- We say that the coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ is *generic* if for any $B, B' \in \mathbf{K}(\mathcal{M})$, any $g \in Aut(\mathcal{M})$, and any $gB \subseteq X \subset_{\mathrm{fin}} M$, if $B \leq B'$ (as $\mathscr{L}$-structures) and $g{\restriction}B$ is an $\mathscr{L}^k_A$-isomorphism of $\mathcal{M}_\xi[B]$ onto $\mathcal{M}_\xi[gB]$, then there is an automorphism $h \in Aut(\mathcal{M})$ such that $g{\restriction}B \subseteq h$, $hB' \cap X = gB$, and which is an $\mathscr{L}^k_A$-isomorphism of $\mathcal{M}_\xi[B']$ onto $\mathcal{M}_\xi[hB']$.
- Suppose $\xi, \xi_0 : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ be colorings, where ξ is generic. We say that ξ *is constrained by* ξ_0 if for every $B \in \mathbf{K}(\mathcal{M})$, there is an automorphism $g \in Aut(\mathcal{M})$ such that $g{\restriction}B$ is an $\mathscr{L}^k_A$-isomorphism of $\mathcal{M}_\xi[B]$ onto $\mathcal{M}_{\xi_0}[gB]$.

Remark 1.12. Suppose $\xi, \xi_0 : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ be colorings. Then ξ is a generic coloring constrained by ξ_0 if and only if $\mathcal{M}_\xi$ is a generic expansion of $\mathcal{M}$ constrained by $\mathcal{M}_{\xi_0}$.

Now that we have formulated what generic colorings and constrained generic colorings actually are, we can use these in formulating a Ramsey property. In fact, we will formulate two of these, but one of these is "temporary," pending the proof of Theorem 1.16, which obviates the weak generic infinitary Ramsey property.

Definition 1.13 (Generic infinitary Ramsey property)**.** Let $\mathbf{K}$ be a finitely-rigid strongly coherent class of finite $\mathscr{L}$-structures with generic model $\mathcal{M}$.

- For $A \in \mathbf{K}$, we say that $\mathcal{M}$ has the *generic infinitary A-Ramsey property (A-GRP)* if for every generic coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$, there is an embedding $e : \mathcal{M} \to \mathcal{M}$ such that ξ is constant on $\mathrm{Emb}(A, e\mathcal{M})$.
- For $A \in \mathbf{K}$, we say that $\mathcal{M}$ has the *weak generic infinitary A-Ramsey property (weak-A-GRP)* if:
  i. For every coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$, there is a generic coloring $\zeta : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ constrained by ξ.
  ii. For every generic coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$, there is an embedding $e : \mathcal{M} \to \mathcal{M}$ such that ξ is constant on $\mathrm{Emb}(A, e\mathcal{M})$.

As usual, $\mathcal{M}$ has the generic infinitary Ramsey property (GRP) if it has A-GRP for every $A \in \mathbf{K}$, and it has the weak generic infinitary Ramsey property (weak-GRP) if it has weak-A-GRP for every $A \in \mathbf{K}$

Although it was not especially satisfying in itself, the amalgamated Ramsey property turns out to have some value in that is a useful intermediate step in dealing the generic infinitary Ramsey property. We now demonstrate the equivalence of the finitary Ramsey property and weak-GRP using just such an intermediate step.

Lemma 1.14. *Let* $\mathbf{K}$ *be a finitely-rigid strongly coherent class with generic model* $\mathcal{M}$*. Let* $A \in \mathbf{K}$*, and let* $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ *be a generic coloring. Suppose there is a* $\mathbf{K}$*-embedding system* $(f_B)_B$ *on* $\mathcal{M}$ *such that for every* $B \in \mathbf{K}(\mathcal{M})$*,* ξ *is constant on* $\mathrm{Emb}(A, f_B[B])$*. Then there is an embedding* $e : \mathcal{M} \to \mathcal{M}$ *such that* ξ *is constant on* $\mathrm{Emb}(A, e[\mathcal{M}])$*.*

Proof. Let $C_0, ..., C_n, ...$ be members of $\mathbf{K}(\mathcal{M})$ such that $C_n \leq C_{n+1}$ for all n and $\bigcup_n C_n = M$. For each n, let $u_n : C_n \to \mathcal{M}$ be an embedding such that ξ is constant on $\mathrm{Emb}(A, u_n C_n)$. By the pigeonhole principle, there is an $i^* \in [k]$ such that $\{n : \xi[\mathrm{Emb}(A, u_n C_n)] = \{i^*\}\}$ is infinite, so without loss of generality, we may assume that $\xi[\mathrm{Emb}(A, u_n C_n)] = \{1\}$ for all n.

Now, since ξ is a generic coloring, we know that for each n, there is an automorphism $h \in Aut(\mathcal{M})$ extending u_n which is an $\mathscr{L}_A^k$-isomorphism of $\mathcal{M}_\xi[C_{n+1}]$ onto $\mathcal{M}_\xi[hC_{n+1}]$; then ξ is still constant on $\mathrm{Emb}(A, hC_{n+1})$. Thus, we may assume that $u_n \subseteq u_{n+1}$ for all n, so $e = \bigcup_n u_n$ is an embedding $\mathcal{M} \to \mathcal{M}$ such that ξ is constant on $\mathrm{Emb}(A, e\mathcal{M})$. $\square$

Proposition 1.15. *Let* $\mathbf{K}$ *be a finitely-rigid strongly coherent class with generic model* $\mathcal{M}$*.*

1. $\mathbf{K}$ *has the Ramsey property.*

2. $\mathcal{M}$ *has the weak generic infinitary Ramsey property.*

Proof of 1⇒2. By Lemma 1.14, part (ii) of weak-GRP is immediate once one has generic coloring and an appropriate $\mathbf{K}$-embedding system in hand, so we need only verify part (i). Let $A \in \mathbf{K}$, and let $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ be an arbitrary coloring. By A-ARP, which we know is equivalent to the A-Ramsey property, let $(f_B)_{B \in \mathbf{K}(\mathcal{M})}$ be a $\mathbf{K}$-embedding system such that ξ is constant on $\mathrm{Emb}(A, f_B B)$ for every $B \in \mathbf{K}(\mathcal{M})$.

Let $C_0, ..., C_n, ...$ be members of $\mathbf{K}(\mathcal{M})$ such that $C_n \leq C_{n+1}$ for all n and $\bigcup_n C_n = M$. By the pigeonhole principle, there is an $i^* \in [k]$ such that $\big\{n : \xi\big[\mathrm{Emb}(A, f_{C_n} C_n)\big] = \{i^*\}\big\}$ is infinite, so without loss of generality, we may assume that $\xi\big[\mathrm{Emb}(A, f_{C_n} C_n)\big] = \{1\}$ for all n. We define $\zeta : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ by setting $\zeta(u) = 1$ for all $u \in \mathrm{Emb}(A, \mathcal{M})$. Then ζ is a generic coloring constrained by ξ. $\square$

Proof of 2⇒1. Given $A \in \mathbf{K}$, we show that $\mathcal{M}$ has the amalgamated A-Ramsey property. So, let $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ be an arbitrary coloring. By part (i) of weak-GRP, there is a generic coloring $\zeta : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ constrained by ξ.

By part (ii) of weak-GRP, there is an elementary embedding $e : \mathcal{M} \to \mathcal{M}$ such that ζ is constant on $\mathrm{Emb}(A, e\mathcal{M})$. To define the appropriate $\mathbf{K}$-embedding system $(f_B)_B$, let $B \in \mathbf{K}(\mathcal{M})$ be given. Since ζ is constrained by ξ, there is an automorphism $g \in Aut(\mathcal{M})$ such that $g{\restriction}eB$ is $\mathscr{L}_A^k$-isomorphism of $\mathcal{M}_\zeta[eB]$ onto $\mathcal{M}_\xi[geB]$, and it follows that ξ is constant on $\mathrm{Emb}(A, geB)$. Thus, we may set $f_B = ge{\restriction}B$. This completes the proof. $\square$

Of course, *if* constrained generic colorings simply always exist (regardless of and without appeal to the Ramsey property), then the word "weak" can be removed from the proposition – see Theorem 1.17 below. It turns out that constrained generic colorings do always exist, but this is a corollary of the following more general fact which we will prove in Section 2.

Theorem 1.16. *Let* $\mathbf{K}$ *be a strongly coherent class of finite* $\mathscr{L}$*-structures, and let* $\mathscr{L}'$ *be an extension of* $\mathscr{L}$ *by finitely many new relation symbols and no function symbols. Let* $\mathcal{M}$ *be the generic model of* $\mathbf{K}$*, and let* $\mathcal{N}$ *be an* $\mathscr{L}'$*-expansion of* $\mathcal{M}$*. Then, there is a generic expansion* $\mathcal{M}'$ *of* $\mathcal{M}$ *constrained by* $\mathcal{N}$*.*

Taking Theorem 1.16 as given, the proof of our characterization of the Ramsey property is now easy to complete.

Theorem 1.17 (Corollary to Theorem 1.16)**.** *Let* $\mathbf{K}$ *be a finitely-rigid strongly coherent class with generic model* $\mathcal{M}$*.*

1. $\mathbf{K}$ *has the Ramsey property.*

2. $\mathcal{M}$ *has the amalgamated Ramsey property.*

3. $\mathcal{M}$ *has the generic infinitary Ramsey property.*

Proof. We have already proven 1⇔2. For 1&2⇒3, we just appeal to Lemma 1.14 in just the same way that we did in the proof of Proposition 1.15(1⇒2).

For 3⇒1&2, we observe that in the proof of Proposition 1.15(2⇒1), we recovered a generic coloring constrained by a given coloring by assumption. In the current scenario, we may use the Theorem 1.16 to produce this constrained generic coloring and then repeat the second paragraph of the argument without change. □

2 Proof of Theorem 1.16

For all of this section, let $\mathscr{L}, \mathscr{L}_1$ be languages subject to our conventions established earlier, where $\mathrm{sig}(\mathscr{L}_1)$ contains $\mathrm{sig}(\mathscr{L})$ but has only finitely many new relation symbols not in $\mathscr{L}$ and no new constant symbols. We also fix a strongly coherent class $\mathbf{K}$, its generic model $\mathcal{M}$, and an $\mathscr{L}_1$-expansion $\mathcal{N}$ of $\mathcal{M}$.

Definition 2.1. We define $\mathsf{Age}_0(\mathcal{N}/\mathcal{M}) = \{\mathcal{N}[A] : A \in \mathbf{K}(\mathcal{M})\}$, which is not $Aut(\mathcal{M})$-closed, and two sets of pairs,

$$\mathbb{P} = \mathbb{P}^{\mathcal{N}} = \{(B, B_0) : B_0, B \in \mathsf{Age}(\mathcal{N}/\mathcal{M}) \text{ and } B_0 \leq B\}$$

$$\mathbb{P}_0 = \mathbb{P}_0^{\mathcal{N}} = \{(B, B_0) : B_0, B \in \mathsf{Age}_0(\mathcal{N}/\mathcal{M}) \text{ and } B_0 \leq B\}$$

Obviously, $\mathbb{P}_0 \subseteq \mathbb{P}$. A pair $(B, B_0) \in \mathbb{P}$ is a structure in the language $\mathscr{L}_1^+$ extending $\mathscr{L}_1$ by a new predicate U naming B_0. Clearly, $Aut(\mathcal{M})$ acts on $\mathbb{P}$ via $g\cdot(B, B_0) = (gB, gB_0)$, but $\mathbb{P}_0$ is not $Aut(\mathcal{M})$-closed.

Now, let $(B, B_0) \in \mathbb{P}_0$. We say that (B, B_0) *is a bad pair (is bad)* if for every $g \in Aut(\mathcal{M})$, if $g{\restriction}B$ is an $\mathscr{L}_1^+$-isomorphism of (B, B_0) onto (gB, gB_0), then $gB \cap \mathrm{acl}_0^{\mathcal{N}}(gB_0) \neq gB_0$, where

$$\mathrm{acl}_0^{\mathcal{N}}(gB_0) := B_0 \cup \left\{a \in M : \mathrm{qftp}^{\mathcal{N}}(a/B_0) \text{ is algebraic in } \mathcal{N}\right\}.$$

In a strong sense bad pairs are the obstruction to $\mathcal{N}$ itself being a generic expansion of $\mathcal{M}$. Thus, to find a generic expansion of $\mathcal{M}$ constrained by $\mathcal{N}$, we just need a method for eliminating bad pairs, and this is the purpose of $\mathcal{N}$-selectors.

Definition 2.2. An $\mathcal{N}$*-selector* is a function $\sigma : \mathsf{Age}(\mathcal{N}/\mathcal{M}) \to \{0, 1\}$ such that:

S1. For all $(B, B_0) \in \mathbb{P}$, if $\sigma(B) = 1$, then $\sigma(B_0) = 1$.

S2. For all $(B, B_0) \in \mathbb{P}_0$, if (B, B_0) is bad, then $\sigma(B) = 0$.

S3. σ is $Aut(\mathcal{M})$-invariant.

S4. For every $A \in \mathbf{K}(\mathcal{M})$, there is some $g \in Aut(\mathcal{M})$ such that $\sigma\big(\mathcal{N}[gA]\big) = 1$.

S5. The class

$$\mathbf{K}^\sigma = \{B \in \mathsf{Age}(\mathcal{N}/\mathcal{M}) : \sigma(B) = 1\} \text{ (closing under isomorphisms)}$$

has disjoint-AP.

Given an $\mathcal{N}$-selector σ, we still need to compile an expansion $\mathcal{M}'$ of $\mathcal{M}$. Since $T_{\mathbf{K}}$-categorical, it is actually enough to recover the theory $Th(\mathcal{M}') \supseteq T_{\mathbf{K}}$. Thus, from σ, we must specify the appropriate theory $T_\sigma \supseteq T_{\mathbf{K}}$, and Definition 2.4 does just this. The key fact about T_σ, which we've just discussed, is the following straightforward observation.

Observation 2.3. Suppose σ is an $\mathcal{N}$-selector. Then T_σ is satisfiable, and every countable model of T_σ amounts to a generic expansion of $\mathcal{M}$ constrained by $\mathcal{N}$.

Definition 2.4. Let $\sigma : \mathsf{Age}(\mathcal{N}/\mathcal{M}) \to \{0,1\}$ be an $\mathcal{N}$-selector.

- For each n, let $K_\sigma(n) = \{B \in \mathsf{Age}(\mathcal{N}/\mathcal{M}) : \sigma(B) = 1,\ |B| = n\}/\!\cong$. We also define sentences

$$\texttt{allow}_{\sigma,n} = \forall x_1 ... x_n \bigvee_{B \in K_\sigma(n)} \theta_B(x_1, ..., x_n)$$

$$\texttt{require}_{\sigma,n} = \bigwedge_{B \in K_\sigma(n)} \exists x_1 ... x_n\, \theta_B(x_1, ..., x_n)$$

- For all $(B, B_0) \in \mathbb{P}$, if $\sigma(B) = 1$, then for each k, let $\texttt{extend}^k_{B,B_0} =$

$$\forall \overline{x} \left(\theta_{B_0}(\overline{x}) \to \exists \overline{y}^1 ... \overline{y}^k \bigwedge_i \theta_B(\overline{x}, \overline{y}^i) \wedge \bigwedge_{i<j} \overline{y}^i \cap \overline{y}^j = \emptyset \right).$$

- We define T_σ to be the theory,

$$\left\{\texttt{require}_{\sigma,n},\, \texttt{allow}_{\sigma,n} : n \in \mathbb{N}\right\} \cup \left\{\texttt{extend}^k_{B,B_0} : (B, B_0) \in \mathbb{P},\ \sigma(B) = 1,\ k \in \mathbb{N}\right\}$$

We have now reduced the question of the existence of generic expansions of $\mathcal{M}$ constrained by $\mathcal{N}$ to the question of the existence of $\mathcal{N}$-selectors. Recovery of an $\mathcal{N}$-selector amounts to making infinitely many decisions that are expected to "cohere" appropriately. Rather than making these decisions explicitly one by one, we find it is much easier to allow certain ultrafilters on $\mathbf{K}(\mathcal{M})$ to make all of these decisions for us. Exactly which utrafilters we should use is expressed in the next two definitions.

Definition 2.5. We define a family of ultrafilters on $\mathbf{K}(\mathcal{M})$.

- We define

$$\mathsf{Cone} = \{X \subseteq \mathbf{K}(\mathcal{M}) : (\exists A \in \mathbf{K}(\mathcal{M}))\, \mathbf{K}_A(\mathcal{M}) \subseteq X\}.$$

  where for each $A \in \mathbf{K}(\mathcal{M})$, $\mathbf{K}_A(\mathcal{M}) = \{B \in \mathbf{K}(\mathcal{M}) : A \leq B\}$. It is not hard to check that Cone is a proper filter on $\mathbf{K}(\mathcal{M})$.

- An ultrafilter $\mathscr{U}$ on $\mathbf{K}(\mathcal{M})$ is a Cone-ultrafilter just in case $\mathsf{Cone} \subseteq \mathscr{U}$, and we write S_{Cone} for set of all Cone-ultrafilters.

Definition 2.6 ($\mathcal{N}$-selecting ultrafilters)**.** Let $A \in \mathbf{K}(\mathcal{M})$.

- We define U^0_A to be the set of those $A' \in \mathbf{K}_A(\mathcal{M})$ such that for every finite set $A \subseteq X \subset_{\text{fin}} M$, there is an automorphism $g \in Aut(\mathcal{M}/A)$ such that $gA' \cap X = A$ and $g\restriction A'$ is an isomorphism of $\mathcal{N}[A']$ onto $\mathcal{N}[gA']$. We then define $U_A = \bigcap_{A_0 \leq A} U^0_{A_0}$.

- Next, we define

$$V_A = \bigcup \big\{U_{gA} : g \in Aut(\mathcal{M}),\ g\restriction A \text{ is an isomorphism of } \mathcal{N}[A] \text{ onto } \mathcal{N}[gA]\big\}.$$

  We observe that $\{V_{gA} : g \in Aut(\mathcal{M})\}$ is a finite set, but its members need not be pairwise disjoint.

- Finally, we define $W_A = \bigcup\{V_{gA} : g \in Aut(\mathcal{M})\}$

Now, we define an *$\mathcal{N}$-selecting ultrafilter* to be just a Cone-ultrafilter $\mathscr{U}$ such that $W_A \in \mathscr{U}$ for every $A \in \mathbf{K}(\mathcal{M})$. Given an $\mathcal{N}$-selecting ultrafilter $\mathscr{U}$, we define a "pre-selector" function $\sigma^{\mathscr{U}}_0 : \mathsf{Age}_0(\mathcal{N}/\mathcal{M}) \to \{0,1\}$ by

$$\sigma^{\mathscr{U}}_0\big(\mathcal{N}[A]\big) = 1 \iff V_A \in \mathscr{U}$$

and then we extend $\sigma^{\mathcal{M}}_0$ to a function $\sigma^{\mathscr{U}} : \mathsf{Age}(\mathcal{N}/\mathcal{M}) \to \{0,1\}$ by setting

$$\sigma^{\mathscr{U}}\big(g{\cdot}\mathcal{N}[A]\big) = \sigma^{\mathscr{U}}_0(\mathcal{N}[A])$$

for all $g \in Aut(\mathcal{M})$.

With these definitions in place, there are two tasks left before us. We must verify the function $\sigma^{\mathscr{U}}$ arising from an $\mathcal{N}$-selecting ultrafilter is actually an $\mathcal{N}$-selector, and we do this in Lemma 2.7. We must also demonstrate that $\mathcal{N}$-selecting ultrafilters actually exist, and this we do in Lemma 2.8. Together, Lemmas 2.7 and 2.8 complete the proof of Theorem 1.16.

Lemma 2.7. *If $\mathscr{U}$ is an $\mathcal{N}$-selecting ultrafilter, then $\sigma^{\mathscr{U}}$ is an $\mathcal{N}$-selector.*

Proof. Requirements S1, S3, and S4 are the most straightforward, so we treat those first. For S1: Let $B, B_0 \in \mathsf{Age}(\mathcal{N}/\mathcal{M})$ such that $B_0 \leq B$. By definition of $\mathsf{Age}(\mathcal{N}/\mathcal{M})$, $B = g{\cdot}\mathcal{N}[A]$ and $B_0 = g{\cdot}\mathcal{N}[A_0]$ for some $A, A_0 \in \mathbf{K}(\mathcal{M})$ and $g \in Aut(\mathcal{M})$. Since $V_A \subseteq V_{A_0}$ and $\mathscr{U}$ is a filter, we have

$$\sigma^{\mathscr{U}}(B) = 1 \Rightarrow V_A \in \mathscr{U} \Rightarrow V_{A_0} \in \mathscr{U} \Rightarrow \sigma^{\mathscr{U}}(B_0) = 1.$$

For S3: Let $B \in \mathsf{Age}(\mathcal{N}/\mathcal{M})$ and $g \in Aut(\mathcal{M})$ be given. Let $A \in \mathbf{K}(\mathcal{M})$ and $g_0 \in Aut(\mathcal{M})$ such that $B = g{\cdot}\mathcal{N}[A]$. Then obviously $gB = gg_0{\cdot}\mathcal{N}[A]$, so $\sigma^{\mathscr{U}}(gB) = \sigma^{\mathscr{U}}_0(\mathcal{N}[A]) = \sigma^{\mathscr{U}}(B)$ by definition.

For S4: Let $A \in \mathbf{K}(\mathcal{M})$ be given. By definition, $W_A \in \mathscr{U}$, and there are $g_1, ..., g_n \in Aut(\mathcal{M})$ such that $W_A = V_{g_1A} \cup \cdots \cup V_{g_nA}$. Since $\mathscr{U}$ is an ultrafilter, $V_{g_iA} \in \mathscr{U}$ for some $i = 1, ..., n$, so that $\sigma^{\mathscr{U}}(\mathcal{N}[g_iA]) = 1$ – as desired.

For S2, suppose (B, B_0) is bad. In particular, suppose $B = h{\cdot}\mathcal{N}[A]$ and $B_0 = h{\cdot}\mathcal{N}[A_0]$, where $A, A_0 \in \mathbf{K}(\mathcal{M})$ and $h \in Aut(\mathcal{M})$. Now, towards a contradiction, suppose $\sigma^{\mathscr{U}}(B) = 1$

– meaning that $V_A \in \mathscr{U}$ and $V_{A_0} \in \mathscr{U}$. Let X be a finite subset of M containing $\mathrm{acl}_0^{\mathcal{N}}(A_0)$. Since $V_{A_0} \in \mathscr{U}$, we may assume that there is an automorphism $g \in Aut(\mathcal{M}/A_0)$ such that $gA \cap X = A_0$ and $g{\restriction}A$ is an isomorphism of $\mathcal{N}[A]$ onto $\mathcal{N}[gA]$. But since (B, B_0) is bad, we know that

$$gA \cap X \supseteq gA \cap \mathrm{acl}_0^{\mathcal{N}}(A_0) \supsetneq A_0$$

a contradiction. Thus, $\sigma^{\mathscr{U}}(B) = 0$. Finally, S5 is immediate from S2 and the definition of $\mathscr{U}$. □

Lemma 2.8. *$\mathcal{N}$-selecting ultrafilters exist.*

Proof. Let $\mathscr{F}_1 =$

$$\left\{X \subseteq \mathbf{K}(\mathcal{M}) : \big(\exists\, n, A_1, ..., A_n, A' \in \mathbf{K}(\mathcal{M})\big)\, \mathbf{K}_{A'}(\mathcal{M}) \cap W_{A_1} \cap \cdots \cap W_{A_n} \subseteq X\right\}.$$

Then, an $\mathcal{N}$-selecting ultrafilter is precisely an ultrafilter $\mathscr{U}$ on $\mathbf{K}(\mathcal{M})$ such that $\mathscr{F}_1 \subseteq \mathscr{U}$, and to prove that $\mathcal{N}$-selecting ultrafilters exist, it is enough to prove that $\mathscr{F}_1$ is a proper filter. To do this, it is enough to prove the following claim.

Claim. For any n and any $A_1, ..., A_n, A' \in \mathbf{K}(\mathcal{M})$, then set $\mathbf{K}_{A'}(\mathcal{M}) \cap W_{A_1} \cap \cdots \cap W_{A_n}$ is infinite.

Proof of claim. We may assume that $\mathrm{acl}^{\mathcal{M}}(A_1 \cup \cdots \cup A_n) \cap A' = \mathrm{acl}^{\mathcal{M}}(\emptyset)$. Let

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k \subseteq \cdots \subset_{\mathrm{fin}} M$$

such that $\bigcup_k X_k = M$, and let $C \in \mathbf{K}(\mathcal{M})$ such that

$$\mathrm{acl}^{\mathcal{M}}(A_1 \cup \cdots \cup A_n \cup A') \subsetneq C.$$

Let $Q = \{\mathcal{N}[gC] : g \in Aut(\mathcal{M}/A')\} \,/{\cong_{A'}}$, a finite set of $\mathscr{L}'$-isomorphism types. We may choose a sequence of automorphisms $g_0, g_1, ..., g_k, ... \in Aut(\mathcal{M}/A')$ such that for each k,

$$g_kC \cap \left(X_k \cap \bigcup_{j<k} g_jC\right) = A'.$$

Now, we define a function $\xi : \mathbb{N} \to Q$ by $\xi(k) = \mathcal{N}[g_kC]/{\cong_{A'}}$. Since Q is finite, by the pigeonhole principle, there is an infinite set $K \subseteq \mathbb{N}$ such that ξ is constant on K. Finally, we observe that

$$\{g_kC : k \in K\} \subseteq \mathbf{K}_{A'}(\mathcal{M}) \cap W_{A_1} \cap \cdots \cap W_{A_n}$$

which completes the proof. □

□

⋆ This completes the proof of Theorem 1.16.

3 Patterning, Ehrenfeucht-Mostowski models, and linear ordering

In this section, we will re-prove a well-known fact about Ramsey classes – that they always (implicitly) carry linear orderings. This fact was proved in [5] using actions using extreme amenability and the fact that the set of all linear orderings of a set M is a compact Hausdorff space. Here, we will take a rather different, more model-theoretic approach, proving Theorem 3.1 below using the Patterning property (i.e. a generalization of the ability to construct of indiscernible sequences) and generalized Ehrenfeucht-Mostowski models. Along the way, we will give a novel proof using our constrained generic expansions technology of the fact (due to [9]) that the Ramsey property and the Patterning property are equivalent. As a corollary to this proof, we find that in the absence of the Ramsey property, one can still recover "generic" pictures that may not be indiscernible.

Theorem 3.1. *Let $\mathbf{K}$ be a finitely-rigid strong Fraïssé class with generic model $\mathcal{M}$, and suppose that $|S_1(T_{\mathbf{K}})| = 1$. If $\mathbf{K}$ has the Ramsey property, then there is an quantifier-free formula $\varphi(x, y)$ of $\mathscr{L}$ that in $\mathcal{M}$ defines a dense linear ordering of the universe M without endpoints.*

For all of this section, we fix a finitely-rigid strong Fraïssé class $\mathbf{K}$ with generic model $\mathcal{M}$, and the language of $\mathbf{K}$ is $\mathscr{L}_0$. We reserve $\mathscr{L}$ for languages of "target" structures, and these are not necessarily subject to the conventions on languages put forth in Section 1. For any arbitrary language $\mathscr{L}$, $\mathfrak{C}$ will denote a "very saturated" $\mathscr{L}$-structure.

3.1 Pictures and the Patterning property

In this subsection, we define several types of pictures of $\mathcal{M}$ in arbitrary structures $\mathfrak{C}$, where a picture of $\mathcal{M}$ is a generalization or analog of an sequence $(a_r)_{r\in\mathbb{Q}}$ of elements of $\mathfrak{C}$. We sketch a slightly novel proof the equivalence of the Patterning and Ramsey properties using generic expansions, and we state a theorem on the existence of generalized Ehrenfeucht-Mostowski models relative any Ramsey class.

Definition 3.2. Let $\mathfrak{C}$ be the "big model" of a complete $\mathscr{L}$-theory with infinite models. We define pictures, indiscernible pictures, and generic pictures:

- A *picture of $\mathcal{M}$ in $\mathfrak{C}$*, $\gamma : \mathcal{M} \to \mathfrak{C}$, is a just an injective mapping of M into (a single sort of) $\mathfrak{C}$.
- A picture $\gamma : \mathcal{M} \to \mathfrak{C}$ of $\mathcal{M}$ in $\mathfrak{C}$ is *indiscernible* if for all $n \in \mathbb{N}$, $a_0, ..., a_{n-1}$ and $b_0, ..., b_{n-1}$ in A,
$$\mathrm{qftp}^{\mathcal{M}}(\overline{a}) = \mathrm{qftp}^{\mathcal{M}}(\overline{b}) \;\Rightarrow\; \mathrm{tp}^{\mathfrak{C}}(\gamma\overline{a}) = \mathrm{tp}^{\mathfrak{C}}(\gamma\overline{b}).$$
(For $\Delta \subseteq \mathscr{L}$, Δ-indiscernible pictures are defined similarly.)

Usually, we will denote indiscernible pictures with the letters I or J instead of γ. One more kind of picture will play a role in our discussion below. Let $\gamma, \gamma_0 : \mathcal{M} \to \mathfrak{C}$ be picture of $\mathcal{M}$ in $\mathfrak{C}$.

- For $\Delta \subset_{\text{fin}} \mathscr{L}$, we make an expanded language $\mathscr{L}^{\Delta}$ with a new relation symbols R_φ of arity n for each $\varphi(x_0, ..., x_{n-1}) \in \Delta$, and we make an $\mathscr{L}^{\Delta}$-expansion $\mathcal{M}_\gamma^{\Delta}$ of $\mathcal{M}$ by setting $R_\varphi^{\mathcal{M}_\gamma^\Delta} = \{\overline{a} \in M^n : \mathfrak{C} \models \varphi(\gamma\overline{a})\}$ for each $\varphi(x_0, ..., x_{n-1}) \in \Delta$.
- We say that γ *is* Δ*-generic* if $\mathcal{M}_\gamma^{\Delta}$ is a generic expansion of $\mathcal{A}$. We say that γ *is* Δ*-generic constrained by* γ_0 if $\mathcal{M}_\gamma^{\Delta}$ is a generic expansion of $\mathcal{A}$ constrained by $\mathcal{M}_{\gamma_0}^{\Delta}$.

Definition 3.3 (Patterning property)**.** Let $\mathfrak{C}$ be the "big model" of a complete $\mathscr{L}$-theory with infinite models.

Let $\gamma : \mathcal{M} \to \mathfrak{C}$ be a picture, and let $I : \mathcal{M} \to \mathfrak{C}$ be an indiscernible picture. We say that I *is patterned on* γ if for every $\Delta \subset \mathscr{L}$, every $n \in \mathbb{N}$, and all $a_0, ..., a_{n-1} \in A$, there is an embedding $f = f_{\Delta,\overline{a}} : \mathcal{M}[\overline{a}] \to \mathcal{M}$ such that

$$\text{tp}_\Delta^{\mathfrak{C}}(I\overline{a}) = \text{tp}_\Delta^{\mathfrak{C}}(\gamma f\overline{a}).$$

Now, we say that $\mathbf{K}$ *has the Patterning property* if for every picture $\gamma : \mathcal{M} \to \mathfrak{C}$, there is an indiscernible picture $I : \mathcal{M} \to \mathfrak{C}$ of $\mathcal{M}$ patterned on γ.

The existence of indiscernible sequences is usually stated (as in [7]) with less precision than is actually required in practice. The existence statement in full precision, but generalized to objects richer than pure linear orders, is the following theorem due to [9].

Theorem 3.4. $\mathbf{K}$ *has the Ramsey property if and only if it has the Patterning property.*

Proof (Sketch). For $\Rightarrow$, suppose $\mathbf{K}$ has the Ramsey property – so that its generic model $\mathcal{M}$ has the generic infinitary Ramsey property. Let $\gamma : \mathcal{M} \to \mathfrak{C}$ be a picture of $\mathcal{M}$ in the "big model" $\mathfrak{C}$ of a complete $\mathscr{L}$-theory with infinite models. We may assume that $\mathscr{L}$ is one-sorted. Then by Theorem 1.16, for each $\Delta \subset_{\text{fin}} \mathscr{L}$, we may choose a Δ-generic picture $\gamma_\Delta : \mathcal{M} \to \mathfrak{C}$ constrained by γ, and moreover, we may assume that for $\Delta_1 \subseteq \Delta_2 \subset_{\text{fin}} \mathscr{L}$, for all $\overline{a} \in A$, $\text{tp}_{\Delta_1}^{\mathfrak{C}}(\gamma_{\Delta_1}\overline{a}) \subseteq \text{tp}_{\Delta_2}^{\mathfrak{C}}(\gamma_{\Delta_2}\overline{a})$.

By GRP (and using finite-rigidity), for each $\Delta \subset_{\text{fin}} \mathscr{L}$, there is an elementary embedding $f_\Delta : \mathcal{M} \to \mathcal{M}$ such that the composition $\gamma_\Delta f_\Delta : \mathcal{M} \to \mathfrak{C}$ is Δ-indiscernible. Let $\mathscr{U}$ be an ultrafilter on $\mathcal{P}_{\text{fin}}(\mathscr{L})$ such that for every $\Delta_0 \subset_{\text{fin}} \mathscr{L}$, $\{\Delta : \Delta_0 \subseteq \Delta\} \in \mathscr{U}$. Let

$$(\mathfrak{C}^*, \mathcal{M}^*, \gamma^*) = \prod\nolimits_\Delta (\mathfrak{C}, \mathcal{M}, \gamma_\Delta) / \mathscr{U}$$

$$(\mathfrak{C}^*, \mathcal{M}^*, I) = \prod\nolimits_\Delta (\mathfrak{C}, \mathcal{M}, \gamma_\Delta f_\Delta) / \mathscr{U}$$

and let $h : \mathcal{M} \to \mathcal{M}^*$ be the diagonal elementary embedding $a \mapsto [\Delta \mapsto a]_{\mathscr{U}}$. Then:

- For every $\Delta \subset_{\text{fin}} \mathscr{L}$, $\gamma^* h : \mathcal{M} \to \mathfrak{C}^*$ is Δ-generic constrained by γ.

- $Ih : \mathcal{M} \to \mathfrak{C}^*$ is indiscernible patterned on γ.

For $\Leftarrow$, suppose $\mathbf{K}$ has the Patterning property. We will show that $\mathcal{M}$ has the amalgamated Ramsey property. Let $A_0 \in \mathbf{K}$, and let $\xi : \mathrm{Emb}(A_0, \mathcal{A}) \to [k]$ be some coloring. We view the identity mapping on A as a picture, $id : \mathcal{A} \to \mathcal{A}_\xi$, of $\mathcal{A}$ in the $\mathscr{L}^k_{A_0}$-expansion $\mathcal{A}_\xi$ of $\mathcal{A}$. By the Patterning property, there is an indiscernible picture $I : \mathcal{A} \to \mathcal{M}$, where $\mathcal{A}_\xi \preceq \mathcal{M}$, patterned on id. By definition of patterning, for each $B \in \mathbf{K}(\mathcal{A})$, there is an embedding $f_B : B \to \mathcal{A}$ such that $\mathrm{tp}^{\mathcal{M}}_\Delta(I\bar{b}) = \mathrm{tp}^{\mathcal{A}_\xi}(\gamma f_B \bar{b})$ where $\bar{b}$ is an enumeration of B and $\Delta = \{R_i(x_0, ..., x_{n-1})\}_{i \in [k]}$. Since I is indiscernible (and by finite-rigidity), it follows that ξ is constant on $\mathrm{Emb}(A_0, f_B B)$, so the system $(f_B)_{B \in \mathbf{K}(\mathcal{A})}$ witnesses ARP for ξ. □

Included in the proof of Theorem 3.4 is the follow result which pertains to all finitely-rigid strong Fraïssé classes, not just Ramsey classes. In some sense, it answers the question, "What sort of 'indiscernibility' can you recover when you don't assume the Ramsey property?"

Corollary 3.5 (To the proof of Theorem 3.4. Here, $\mathbf{K}$ need not have the Ramsey property)**.** *Let $\gamma : \mathcal{M} \to \mathfrak{C}$ be a picture of a $\mathcal{M}$ in the "big model" $\mathfrak{C}$ of a complete $\mathscr{L}$-theory with infinite models. Then there is a picture $\gamma' : \mathcal{M} \to \mathfrak{C}$ such that for every $\Delta \subset_{fin} \mathscr{L}$, γ' is Δ-generic constrained by γ.*

To conclude this subsection, we observe that the the classical form of the following theorem actually has nothing to do with linear orders *per se*. That is, the proof of Lemma 5.2.4 of [7] uses quantifier-free-complete types, but does not need these to be in the language of orders.

Theorem 3.6 (Generalized Ehrenfeucht-Mostowski Theorem)**.** *Assume $\mathbf{K}$ has the Ramsey property. Let $\mathfrak{C}$ be the "big model" of a complete $\mathscr{L}$-theory with infinite models and built-in Skolem functions, and let $I : \mathcal{M} \to \mathfrak{C}$ be an indiscernible picture. If $\mathcal{N} \prec \mathfrak{C}$ is the Skolen hull of $I[M]$, then for every automorphism $g \in Aut(\mathcal{M})$, the map $IgI^{-1} : I[M] \to I[M]$ extends to an automorphism of $\mathcal{N}$.*

3.2 Proof of Theorem 3.1

In this subsection, we will use Theorems 3.4 and 3.6 in a new proof of the fact that the generic model $\mathcal{M}$ of a Ramsey class is necessarily linearly ordered by a quantifier-free formula $\varphi(x, y)$. The key lemma is the following, which allows us to "locally order" $\mathcal{M}$ in a definable way.

Lemma 3.7. *Assume $\mathbf{K}$ is a Ramsey class. Let $p(x, y) \in S_2(T_{\mathbf{K}})$ such that $p(x, y) \vDash x \neq y$. Then $p(x, y) \cup p(y, x)$ is inconsistent.*

Proof. Let $p(x, y)$ be as hypothesized, and towards a contradiction, suppose $p(x, y) \cup p(y, x)$ is consistent. Since $T_{\mathbf{K}}$ is $\aleph_0$-categorical, we may choose $a_0, a_1 \in M$ such that $\mathcal{M} \vDash p(a_0, a_1)$ and $\mathcal{M} \vDash p(a_1, a_0)$. Let $\gamma : M \to \mathbb{Q}$ be any bijection. Viewing $\mathbb{Q}$ as a model of DLO, γ is a picture of $\mathcal{A}$ in $\mathbb{Q}$. By Theorems 3.4 and 3.6 and since DLO is $\aleph_0$-categorica, there is an indiscernible picture $I : \mathcal{M} \to \mathbb{Q}$ patterned on γ such that for every $g \in Aut(\mathcal{A})$, IgI^{-1} extends to an automorphism of $\mathbb{Q}$. Without loss of generality, we assume that $I(a_0) < I(a_1)$.

Since $a_0a_1 \equiv a_1a_0$ and using $\aleph_0$-categoricity again, there is an automorphism $g \in Aut(\mathcal{M})$ such that $g(a_0) = a_1$ and $g(a_1) = a_0$. Let $h \in Aut(\mathbb{Q})$ be an extension of IgI^{-1}. Then we find that

$$h(I(a_0)) = (IgI^{-1})(I(a_0)) = I(g(a_0)) = I(a_1),$$
$$h(I(a_1)) = (IgI^{-1})(I(a_1)) = I(g(a_1)) = I(a_0).$$

Thus, $I(a_0) < I(a_1)$ but $h(I(a_0)) > h(I(a_1))$ – contradicting the fact that h is an automorphism of $\mathbb{Q}$. $\square$

To complete the proof of Theorem 3.1, we just need to bootstrap from our "local order" condition to a global ordering.

Proof of Theorem 3.1. Let W be the set of all types $p(x, y) \in S_2(T_{\mathbf{K}})$ such that $p(x, y) \models x{\neq}y$, and let $q_\bullet : W \to W : p \mapsto q_p$ such that for each $p \in W$, $q_p(y, x) = p(x, y)$. Noting that $q_{q_p} = p$ for each $p \in W$, we observe that $q_\bullet$ induces a partition $\pi = \{\{p, q_p\} : p \in W\}$ of W into 2-element sets. Let U denote the set of choice functions $p_\bullet : \pi \to W : C \mapsto p_C(x, y)$ – so that $p_C(x, y) \in C$ for each $C \in \pi$. We observe that if $(a_i)_{i<\omega}$ is an indiscernible sequence in $\mathcal{A}$, then there are $C \in \pi$ and $p_\bullet \in U$ such that $i < j \Leftrightarrow \mathcal{M} \models p_C(a_i, a_j)$.

For an integer $n \geq 2$ and non-empty $U_0 \subseteq U$, let $\theta^n_{U_0}(x_0, ..., x_{n-1})$ be the formula

$$\bigvee_{p_\bullet \in U_0} \left(\left(\bigwedge_{i<j} \bigvee_{C\in\pi} p_C(x_i, x_j) \right) \wedge \left(\bigwedge_{i\not<j} \bigwedge_{C\in\pi} \neg p_C(x_i, x_j) \right) \right)$$

asserting that for some $p_\bullet \in U_0$, $\varphi_{p_\bullet}(x, y) = \bigvee_{C\in\pi} p_C(x, y)$ defines the linear ordering $x_0 < \cdots < x_{n-1}$. Now, let $c_0, c_1, ..., c_i, ...$ be new constant symbols, and let Ψ be the set of sentences

$$T_{\mathbf{K}} \cup \{c_i{\neq}c_j\}_{i<j} \cup \left\{ \neg\theta^n_{U_0}(c_{i_0}, ..., c_{i_{n-1}}) : \begin{array}{l} n \geq 2,\ i_0 < \cdots < i_{n-1}, \\ \emptyset \neq U_0 \subseteq U \end{array} \right\}$$

If Ψ were consistent, then it would have a model in which the interpretations of $c_0, c_1, ..., c_i, ...$ form an indiscernible sequence, so we know that Ψ is inconsistent. In particular, there is a number $N \geq 2$ such that for any $n \geq N$, $T_{\mathbf{K}}$ implies,

$$\forall x_0...x_{n-1} \left(\bigwedge_{i<j} x_i{\neq}x_j \to \bigvee_{\sigma\in Sym(n)} \bigvee_{\emptyset\neq U_0\subseteq U} \theta^n_{U_0}\left(x_{\sigma(0)}, ..., x_{\sigma(n-1)}\right) \right).$$

Now, let $a_0, a_1, ..., a_i, ...$ be any bijective enumeration of $\mathcal{M}$, and for each n, let $A_n = \{a_0, ..., a_{n-1}\}$. For each $n \geq N$, let $\sigma_n \in Sym(n)$ and $U_n \subseteq U$, minimal non-empty for the condition $\mathcal{M} \models \theta^n_{U_n}(a_{\sigma_n(0)}, ..., a_{\sigma_n(n-1)})$. As U is a finite set, by the pigeonhole principle, there is a non-empty $V \subseteq U$ such that $\{n : U_n = V\}$ is infinite. Hence, $T_{\mathbf{K}}$ implies

$$\forall x_0...x_{n-1} \left(\text{“}\theta^n_V \text{ defines a linear ordering of } \{x_0, ..., x_{n-1}\}\text{”}\right).$$

It follows that $\varphi_V(x, y) = \bigvee_{p_\bullet\in V} \varphi_{p_\bullet}(x, y)$ defines a linear ordering of $\mathcal{M}$. $\square$

4 Extreme amenability and Stone spaces

To start of this section, we quote a seminal result of [5] which characterizes the Ramsey property for a Fraïssé class $\mathbf{K}$ in terms of a fixed-point property of the automorphism group $Aut(\mathcal{M})$ of its generic model.

Theorem 4.1. *Let $\mathbf{K}$ be a Fraïssé class with generic model $\mathcal{M}$. Then $\mathbf{K}$ has the Ramsey property if and only if $Aut(\mathcal{M})$ is extremely amenable: Every continuous action of $Aut(\mathcal{M})$ on a compact Hausdorff space has a fixed point.*

The original proof of this theorem proceeds by reducing the question to continuous actions on compact subsets of Euclidean spaces, and broadly speaking, the arguments are very analytic in flavor. In [1], another very analytic argument is given showing that in order to verify extreme amenability of a Polish group G, it is enough to verify that every continuous action of G on a Cantor space has a fixed point. Here, we investigate a weakening of the result of [1], but our approach is (we think) quite different in flavor.

Theorem 4.2. *Let $\mathbf{K}$ be a finitely-rigid strong Fraïssé class with generic model $\mathcal{M}$. $Aut(\mathcal{M})$ is extremely amenable if and only if $Aut(\mathcal{M})$ is extremely amenable relative to Stone spaces: Every continuous action of $Aut(\mathcal{M})$ on a Stone space has a fixed point.*

Our proof of this fact arises from (re)considering the following factorization of the headline equivalence:

$$\text{EA/Stone} \Longrightarrow \text{GRP} \Longleftrightarrow \text{RP} \Longrightarrow \text{EA} \Longrightarrow \text{EA/Stone}$$

where EA stands for "$Aut(\mathcal{M})$ is extremely amenable," EA/Stone stands for "$Aut(\mathcal{M})$ is extremely amenable relative to Stone spaces," and so forth. Then, our contribution in this section amounts to the following:

- In Subsection 4.1, we give a direct proof of EA/Stone $\Rightarrow$ GRP.
- In Subsection 4.2, we prove that EA/Stone $\Rightarrow$ EA by extracting a sufficiently representative action on a Stone space from a given continuous action on an arbitrary compact Hausdorff space.
- In Subsection 4.3, we give a model-theoretic formulation of the proof that RP $\Rightarrow$ EA/Stone – model-theoretic in the sense that it all boils down to showing that, after specifying the right structure, a certain partial type is satisfiable.

4.1 Proof of EA/Stone $\Rightarrow$GRP

The novel portion of this first argument is just the following:

Proposition 4.3. *Let $\mathbf{K}$ be a finitely-rigid strongly coherent class with generic model $\mathcal{M}$. If $Aut(\mathcal{M})$ is extremely amenable relative to Stone spaces, then $\mathbf{K}$ has the generic infinitary Ramsey Property.*

Proof. Let $A \in \mathbf{K}$ and $k \in \mathbb{N}^+$, and let $\xi : \mathrm{Emb}(A, \mathcal{M}) \to [k]$ be a generic coloring. We continue to view $\mathsf{Age}(\mathcal{M}_\xi/\mathcal{M})$ as a set of $\mathscr{L}_A^k$-expansions of finite induced substructures of $\mathcal{M}$; that is, we do not close $\mathsf{Age}(\mathcal{M}_\xi/\mathcal{M})$ under isomorphisms. Let $\mathbb{X}_\xi$ be the set of $\mathscr{L}_A^k$-expansions of $\mathcal{M}$ such that $\mathcal{M}'[B] \in \mathsf{Age}(\mathcal{M}_\xi/\mathcal{M})$ for every $B \in \mathbf{K}(\mathcal{M})$. For each $B \in \mathsf{Age}(\mathcal{M}_\xi/\mathcal{M})$, let $[B]$ denote the set of all $\mathcal{M}' \in \mathbb{X}_\xi$ that have B as an induced substructure. Then, $\mathbb{X}_\xi$ is the Stone space of the boolean algebra generated by $\{[B] : B \in \mathsf{Age}(\mathcal{M}_\xi/\mathcal{M})\}$, and the natural action $Aut(\mathcal{M}) \curvearrowright \mathbb{X}_\xi$ is continuous. Since $Aut(\mathcal{M})$ is extremely amenable relative to Stone spaces, this action has a fixed point, say $\mathcal{M}_1$. That is, $g\mathcal{M}_1 = \mathcal{M}_1$ for all $g \in Aut(\mathcal{M})$.

Assuming $A = \{a_1, ..., a_n\}$, we claim that for some $1 \leq i \leq k$, $\mathcal{M}_1 \vDash R_i(e(a_1), ..., e(a_n))$ for every embedding $e : A \to \mathcal{M}$. Let $1 \leq i \leq k$ and $e, e' \in \mathrm{Emb}(A, \mathcal{M})$ be given, and suppose that $\mathcal{M}_1 \vDash R_i(e(a_1), ..., e(a_n))$. Since $\mathcal{M}$ is $\mathbf{K}$-homogeneous, there is an automorphism $g \in Aut(\mathcal{M})$ such that $ge = e'$, and it follows that $\mathcal{M}_1 \vDash R_i(ge(a_1), ..., ge(a_n))$ because $g\mathcal{M}_1 = \mathcal{M}_1$, so that $\mathcal{M}_1 \vDash R_i(e'(a_1), ..., e'(a_n))$.

Since $\mathcal{M}_1 \in \mathbb{X}_\xi$, we can generate a $\mathbf{K}$-embedding system $(f_B)_B$ such that $\xi(f_B u) = i$ for all $B \in \mathbf{K}(\mathcal{M})$ and $u \in \mathrm{Emb}(A, B)$. Thereafter, we use Lemma 1.14 to produce an elementary embedding $f : \mathcal{M} \to \mathcal{M}$ such that ξ is constant on $\mathrm{Emb}(A, f\mathcal{M})$ – as desired. □

4.2 Direct proof of EA/Stone ⇒EA

As promised, in this subsection, we will prove that EA/Stone ⇒EA without appealing to previously known results. Thus, we must show directly how to "transform" fixed points from the context of extreme amenability relative to Stone spaces into fixed points from the context of extreme amenability for arbitrary compact Hausdorff spaces. An important tool in carrying out this transform is Lemma 4.5 below, but to set up this lemma, we need a few definitions and bits of notation as follows.

Definition 4.4. Let X be a topological space.

- $\mathsf{Op}(X)$ and $\mathsf{Cl}(X)$ denote the sets of open and closed sets of X, respectively.
- $\mathbf{B}_X^0$ is the boolean sub-algebra of $\mathcal{P}(X)$ generated by $\mathsf{Cl}(X)$, and θ is the congruence on $\mathbf{B}_X^0$ generated by,
$$\{(A, \overline{A}) : A \in \mathbf{B}_X^0\}$$
where $\overline{A} = \bigcap\{K \in \mathsf{Cl}(X) : A \subseteq K\}$ is the closure of $A \subseteq X$.
- $\mathbf{B}_X$ is the quotient algebra $\mathbf{B}_X^0/\theta$, and $\mathbb{S}_X = S(\mathbf{B}_X)$ is the Stone space of $\mathbf{B}_X$.

Lemma 4.5. *Let G be a topological group, and let $\alpha : G \times X \to X$ be a continuous action on a topological space X. Define a map,*

$$B_\alpha : G \times \mathbb{S}_X \to \mathcal{P}(\mathbb{S}_X)$$

$$B_\alpha(g, p) = \left\{ q \in \mathbb{S}_X : \bigwedge_{t=0,1} \left\{ (\alpha_g K)/\theta : \begin{array}{c} p(K/\theta) = t, \\ K \in \mathsf{Cl}(X) \end{array} \right\} \subseteq q^{-1}(t) \right\}.$$

Then:

1. *For all $p \in \mathbb{S}_X$ and $g \in G$, $|B_\alpha(g,p)| = 1$, so abusing the notation, we may redefine B_α to be a function $G \times \mathbb{S}_X \to \mathbb{S}_X$*

2. *With this redefinition, B_α is an action $G \times \mathbb{S}_X \to \mathbb{S}_X$.*

3. *If X is compact, then as an action $G \times \mathbb{S}_X \to \mathbb{S}_X$, B_α is continuous.*

(We observe that in item 3, local compactness of X would be sufficient, but that discussion is not needed here.)

Proof. Given $g \in G$ and $p \in \mathbb{S}_X$, we show first that $|B_\alpha(g,p)| \leq 1$. Suppose $q, q' \in B_\alpha(g,p)$. Then, suppose $A/\theta \in \mathbf{B}_X$ is such that $q(A/\theta) = 1$. Since α_g, $\alpha_{g^{-1}}$ are homeomorphisms of X and $\alpha_g \alpha_{g^{-1}} = id$, we know that $\alpha_{g^{-1}} \left[\overline{A}\right]$ is closed and

$$A/\theta = \overline{A}/\theta = \left(\alpha_g \left[\alpha_{g^{-1}} \left[\overline{A}\right]\right]\right)/\theta.$$

If $p(\alpha_{g^{-1}} \left[\overline{A}/\theta\right]) = 0$, we'd have $q(A/\theta) = 0$ by definition of B_α, so it needs be that $p(\alpha_{g^{-1}} \left[\overline{A}/\theta\right]) = 1$ – which implies that $q'(A/\theta) = 1$. By symmetry, we've shown that $q = q'$, as required.

Next, we will show that $B_\alpha(g,p)$ is non-empty. For this, we must show that for closed subsets $K_0, ..., K_{m-1}$ and $K'_0, ..., K'_{n-1}$ of X, if $p(K_i/\theta) = 1$ and $p(K'_j/\theta) = 0$ for all $i < m$, $j < n$, then

$$\{((\alpha_g K_i)/\theta, 1) : i < m\} \cup \left\{\left((\alpha_g K'_j)/\theta, 0\right) : j < n\right\}$$

extends to a boolean algebra homomorphism $\langle \mathbb{K} \rangle_{\mathbf{B}_X} \to 2$, where $\langle \mathbb{K} \rangle_{\mathbf{B}_X}$ is the sub-algebra of $\mathbf{B}_X$ generated by $\mathbb{K} = \left\{\alpha_g K_i/\theta, \alpha_g K'_j/\theta\right\}_{i,j}$. If not, then there must be a set $W \subseteq X$ such that

$$\bigcap_{i<n} \alpha_g K_i \cap \bigcap_{j<m} X \setminus \alpha_g K'_j \subseteq \partial W$$

where as usual $\partial W = \overline{W} \cap \overline{(X \setminus \overline{W})}$ is the boundary of W.[1] Since α_g, $\alpha_{g^{-1}}$ are homeomorphisms of X and $\alpha_g \alpha_{g^{-1}} = id$, it follows that

$$\bigcap_{i<n} K_i \cap \bigcap_{j<m} X \setminus K'_j \subseteq \partial(\alpha_{g^{-1}} W)$$

and this contradicts the assumption that $p(K_i/\theta) = 1$ and $p(K'_j/\theta) = 0$ for all $i < m$ and $j < n$. This completes the demonstration of the fact that $B_\alpha(g,p)$ is non-empty. Thus, $|B_\alpha(g,p)| = 1$ as required by item 1 of the lemma.

[1]One observes that θ identifies all boundaries with $\emptyset$ and nothing else of consequence. Strictly speaking, θ identifies positive boolean combinations of boundaries with $\emptyset$, but dealing with this technicality does not meaningfully alter the argument – except making it a little nastier.

For item 2, it is a routine exercise to show that B_α is an action, so we omit this. We move, then, to proving that B_α is continuous when X is compact. For this, we consider a basic clopen set $[K/\theta]$ of $\mathbb{S}_X$, and we must show that

$$B_\alpha^{-1}[K/\theta] = \{(g,p) \in G \times \mathbb{S}_X : B_\alpha(g,p) \in [K/\theta]\}$$

is open. Clearly, $B_\alpha^{-1}[K/\theta] = \{(g,p) : p \in [\alpha_{g^{-1}} K/\theta]\}$. Then, if we view the action α as a topological morphism from G to the homeomorphism group $\mathrm{Homeo}(X)$ of X – viewed as a topological group with the compact-open topology – then we have

$$B_\alpha^{-1}[K/\theta] = \bigcup \left\{ U(K', K) \times [K'/\theta] : K' \in \mathsf{Cl}(X) \right\}$$

where

$$U(K', K) = \left\{ h \in \mathrm{Homeo}(X) : h[K'] \subseteq X \setminus \overline{(X \setminus K)} \right\}$$

is a basic open set of $\mathrm{Homeo}(X)$. This completes the proof of the fact that B_α is continuous provided that X is compact. $\square$

Given Lemma 4.5 and the obvious implication EA $\Rightarrow$ EA/Stone, we complete this proof of Theorem 4.2 by proving the following proposition.

Proposition 4.6. *Let G be a topological group. If G is extremely amenable relative to Stone spaces, then G is extremely amenable.*

Proof. Suppose G is extremely amenable relative to Stone spaces. Further, suppose that X is a compact Hausdorff space – not necessarily a Stone space – and that $\alpha : G \times X \to X$ is a continuous action. Let $B_\alpha : G \times \mathbb{S}_X \to \mathbb{S}_X$ be the induced action as defined in Lemma 4.5-1,2. By Lemma 4.5-3, since X is compact, B_α is continuous, so applying extreme amenability relative to Stone spaces, let $p^* \in \mathbb{S}_X$ be a fixed-point of B_α.

It is not hard to see that $\{K \in \mathsf{Cl}(X) : p^*(K/\theta) = 1\}$ extends uniquely to an ultrafilter $\mathscr{U}$ of subsets of X. As X is compact Hausdorff, $\mathscr{U}$ has a unique limit, x^*, and this is clearly a fixed-point of α. $\square$

4.3 Proof of RP$\Rightarrow$EA

In this subsection, we will prove – using extreme amenability relative to Stone spaces – that the Ramsey Property is sufficient for extreme amenability. That is, we prove:

Proposition 4.7. *Let $\mathbf{K}$ be a finitely-rigid strongly coherent class with generic model $\mathcal{M}$. If $\mathbf{K}$ has the Ramsey Property, then $Aut(\mathcal{M})$ is extremely amenable.*

For the rest of this subsection, we fix a Ramsey class $\mathbf{K}$ with generic model $\mathcal{M}$, and we set $G = Aut(\mathcal{M})$.

Since we have already proven EA/Stone$\Rightarrow$EA, it is enough to prove that $Aut(\mathcal{M})$ is extremely amenable relative to Stone spaces. Given a boolean algebra $\mathbf{B}$ and a continuous action $\psi : G \times S(\mathbf{B}) \to S(\mathbf{B})$, we will define a structure $\mathcal{N} = \mathcal{N}_{\mathbf{B},\psi}$ and a partial type $\Sigma(x)$

over $\mathcal{N}$ such that if $\mathcal{N} \preceq \mathcal{N}'$ and q^* is a realization of $\Sigma(x)$ in $\mathcal{N}'$, then q^* induces a fixed point of ψ. In this conversation, we also assume that $S(\mathbf{B})$ has a dense G-orbit under this action. Let $\mathcal{N} = \mathcal{N}_{\mathbf{B},\psi}$ be a two-sorted structure as follows:

- One sort is $S(\mathbf{B})$ itself, $\mathsf{S}^{\mathcal{N}} = S(\mathbf{B})$, and for each $g \in G$, $\mathbf{f}_g^{\mathcal{N}} : \mathsf{S}^{\mathcal{N}} \to \mathsf{S}^{\mathcal{N}}$ is a function symbol interpreted as ψ_g.
- The second sort is denoted B, and it carries the signature of boolean algebras. In $\mathcal{N}$, we have $\mathsf{B}^{\mathcal{N}} = \mathbf{B}$.
- There is also a relation symbol ε of sort $\mathsf{S} \times \mathsf{B}$ interpreted, naturally enough, as the membership relation – i.e. $\varepsilon^{\mathcal{N}} = \{(p, b) : p(b) = 1\}$, where $p \in S(\mathbf{B})$ are viewed as homomorphisms from $\mathbf{B}$ to the 2-element boolean algebra.

Having defined the structure $\mathcal{N}$, we define the appropriate partial type $\Sigma(x) = \Sigma_{\mathbf{B},\psi}(x)$. For $F \subset_{\text{fin}} G$, $n \in \mathbb{N}$, and $b_0, ..., b_{n-1} \in \mathbf{B}$, let $\varphi_F^n(x, \bar{b})$ be the following formula,

$$(\forall z \in \mathsf{S}) \left(\bigvee_{i<n} z \,\varepsilon\, b_i\right) \;\to\; \bigvee_{i<n}\bigwedge_{g\in F}\mathbf{f}_g(x) \,\varepsilon\, b_i$$

asserting that, "if $b_0, ..., b_{n-1}$ form a cover of $S(\mathbf{B})$, then for some $i < n$, $\psi_g(x) \in b_i$ for *every* $g \in F$." Finally, we define our partial type,

$$\Sigma(x) = \left\{ \varphi_F^n(x, \bar{b}) : \begin{array}{l} F \subset_{\text{fin}} G,\, n \in \mathbb{N}, \\ b_0, ..., b_{n-1} \in \mathbf{B} \end{array} \right\}$$

over $\mathcal{N}$. By way of model-theoretic compactness and the following observation, this construction reduces the proof of EA/Stone to proving that that $\Sigma_{\mathbf{B},\psi}(x)$ is finitely satisfiable.

Observation 4.8. Let $\mathbf{B}$ be a boolean algebra, and let $\psi : G \times S(\mathbf{B}) \to S(\mathbf{B})$ be a continuous action. Suppose q^* is a realization of $\Sigma_{\mathbf{B},\psi}$ in some elementary extension $\mathcal{N}'$ of $\mathcal{N}_{\mathbf{B},\psi}$, and define $p^* : \mathbf{B} \to 2$ defined by,

$$p^*(b) = \begin{cases} 1 & \text{if } \mathcal{N}' \models q^* \,\varepsilon\, b \\ 0 & \text{if } \mathcal{N}' \not\models q^* \,\varepsilon\, b. \end{cases}$$

Then $p^* \in S(\mathbf{B})$ is a fixed point of ψ.

At last, to complete the proof of RP⇒EA/Stone⇒EA, we need only prove the following lemma (in which we adapt the proof of a statement from [5]).

Lemma 4.9 (cf. Proposition 4.2 of [5]). *Suppose* $\mathbf{K}$ *has the Ramsey property. Let* $\mathbf{B}$ *be a boolean algebra, and let* $\psi : G \times S(\mathbf{B}) \to S(\mathbf{B})$ *be a continuous action. If* $\mathcal{M}$ *has the amalgamated Ramsey property, then* $\Sigma_{\mathbf{B},\psi}(x)$ *is finitely satisfiable.*

Proof. It is enough to show that for any $F \subset_{\text{fin}} G$, $n \in \mathbb{N}$, and $b_0, ..., b_{n-1} \in \mathbf{B}$ such that $b_0 \vee \cdots \vee b_{n-1} = 1$, $\varphi_F^n(x, \bar{b})$ is consistent. Let $A \in \mathbf{K}(\mathcal{M})$, and let $p_0 \in S(\mathbf{B})$ be any member of the dense G-orbit. Let $B_A = \text{acl}\left(\bigcup_{g\in F} gA\right)$. We observe that

$$X_\bullet : \text{Emb}(A, \mathcal{M}) \to G/G_A : e \mapsto X_e = \{g \in G : e \subset g\}$$

is a bijection, so we may define coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to \{0, 1, ..., n-1\}$ via

$$\xi(e) = \min\{i : (\exists\, g \in X_e)\, \mathcal{N} \models \mathbf{f}_g(p_0)\, \varepsilon\, b_i\}\,.$$

(ξ is well-defined because since $b_0 \vee \cdots \vee b_{n-1} = 1$.) By the amalgamated Ramsey property, there is a $\mathbf{K}$-embedding system $(f_B)_B$ such that ξ is constant on $\mathrm{Emb}(A, f_B B)$ for each $B \in \mathbf{K}(\mathcal{M})$. In particular, ξ is constant on $\mathrm{Emb}(A, f_{B_A} B_A)$ – say that $\xi(e) = i$ for each embedding $e : A \to f_{B_A} B_A$. Since $B_0 \in \mathbf{K}(\mathcal{M})$, there is an automorphism $h_A \in G$ such that $f_{B_A} \subset h_A$. Then, for every $g \in F$, there is a $g' \in G$ such that $g' \restriction A = h_A g \restriction A$ and $\mathcal{N} \models \mathbf{f}_{g'}(p_0)\, \varepsilon\, b_i$. Finally, since G acts on itself, both on the left and the right, continuously, transitively, and faithfully, we can choose $k_A \in G$ such that $h_A g = g k_A$ for each $g \in G$.

Now, if we choose $A_0, ..., A_t, ... \in \mathbf{K}(\mathcal{M})$ such that $\bigcup_t A_t = \mathcal{M}$ and then choose k_{A_t} as in the previous paragraph for each t, then we find ourselves with an automorphism $k \in G$ such that $\mathcal{N} \models \mathbf{f}_{gk}(p_0)\, \varepsilon\, b_i$ for each $g \in F$. Finally, we set $p_1 = \psi_k(p_0)$, we find that $\mathcal{N} \models \mathbf{f}_g(p_1)\, \varepsilon\, b_i$ for each $g \in F$ – as desired. □

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