Dimension and simplicity for Ramsey-expandable classes of finite structures

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September 26, 2016

*** DRAFT ***

Abstract

We introduce a notion of *manufactured dimension function* (with values in $\mathbb{R}\cup\{-\infty\}$) for the generic model of a Ramsey-expandable class of finite structures. This idea allows us to extract an automorphism-invariant dimension from a Hrushovski-style psuedo-finite dimension function as presented in [12, 13]. Adapting the technique of [6], we demonstrate that the existence of a manufactured dimension function whose image is well-ordered is sufficient for super-simplicity. Further, we show that the generic model of a Ramsey-expandable class whose generic theory has the finite sub-model property *always* admits a manufactured dimension function (possibly not well-ordered). Finally, we formulate some conjectures around well-ordering and simplicity of the generic theory of a Ramsey-expandable class that suggest a possible approach to showing that the theory of the Henson graph is not pseudo-finite.

Introduction

In an ultraproduct of finite structures, every definable set X has a non-standard size $|X|$, and following [12, 13], one then finds that the non-standard quantities $\log|X|$ (or the mapping $X \mapsto \log|X|$) have many of the properties of dimension that one might hope for. With some normalization, this leads to the notion of the *fine pseudo-finite dimension*, denoted $\delta(X)$ for the moment. Closely related ideas were used in [11] to study approximate subgroups of simple linear groups, among other things.

In [6], it is shown that in a true ultraproduct, δ detects dividing in the sense that if $\varphi(\overline{x},\overline{b}) \subseteq \psi(\overline{x},\overline{a})$ and $\varphi(\overline{x},\overline{b})$ divides over $\overline{a}$, then *for some* $\overline{b}' \equiv_{\overline{a}} \overline{b}$, $\delta(\varphi(\overline{x},\overline{b}')) < \delta(\psi(\overline{x},\overline{a}))$. In general, pseudo-finite dimensions are not definable, invariant, or continuous (in the terminology of [12]) unless one enrichs the logic to include something like cardinality-comparison quantifiers, so "detecting dividing" is, at least, nearly optimal unless some strong additional assumptions are made. An extensive investigation is made in [5] of just what these additional assumptions might look like if one hopes to connect a pseudo-finite dimension directly to a more traditional model-theoretic notion like simplicity. For theories that are not themselves pseudo-finite, [7] shows that, given model-completeness and a countable language, one can embed the countable model into a large fragment of an ultraproduct in such way as to pull back the pseudo-finite dimension and continue to detect dividing in the sense described above.

In this paper, we investigate pseudo-finite dimensions in a special case of the context studied in [7]. In particular, we define a notion of *manufactured dimension function* for the $\aleph_0$-categorical generic model $\mathcal{M}$ of an amalgamation class $\mathbf{K}$ of finite structures. These are extracted from the pseudo-finite dimensions associated with certain ultraproducts over $\mathbf{K}$. In our formulation, these manufactured dimension functions are $Aut(\mathcal{M})$-invariant, so the notion is more restrictive than either the fine pseudo-finite dimension or the quasi-dimension of [7]. We prove (following [6] very closely) that a manufactured dimension function d detects dividing in the stronger sense that if $\varphi(\overline{x},\overline{b}) \subseteq \psi(\overline{x},\overline{a})$ and $\varphi(\overline{x},\overline{b})$ divides over $\overline{a}$, then $d(\varphi(\overline{x},\overline{b})) < d(\psi(\overline{x},\overline{a}))$. This is more akin to the "dividing detection" situation for MS-measurable theories as in [4].

Just as for the properties investigated in [5], the question remains of whether manufactured dimension functions actually exist. Answering positively, we show that if $\mathbf{K}$ is Ramsey-expandable (has a Ramsey-lift in the terminology of [14]) and $Th(\mathcal{M})$ has the finite sub-model property, then $\mathcal{M}$ does admit manufactured dimension functions. Combining this existence result with the analysis of manufactured dimension functions described above, we notice that this suggests an approach to proving that the Henson graph is not pseudo-finite. That is, since the class of finite triangle-free graphs is Ramsey-expandable and the Henson graph is rosy, if the Henson graph were pseudo-finite, then (we conjecture) it would have to be super-simple.

Outline of the paper.

In Section 1, we collect the necessary definitions and notation for the rest of paper. In particular, we present the basic construction of fine pseudo-finite dimensions, denoted $d^{\mathscr{U}}_{\mathrm{Hru}}$, and we present basic facts and notation around amalgamation classes of finite structures, the finite sub-model property for generic theories of such classes, and the Ramsey property for such classes.

In Section 2, we define what we mean by a *manufactured dimension function* for the generic model of a so-called strongly coherent class. We prove the "dividing detection" result for manufactured dimension functions, and we demonstrate that a certain well-ordering condition on the image of a manufactured dimension function is sufficient for super-simplicity.

In Section 3, we prove that if $\mathcal{M}$ is the generic model of a Ramsey-expandable class $\mathbf{K}$ and $Th(\mathcal{M})$ has the finite sub-model property (equivalently, $\mathbf{K}$ is tame), then $\mathcal{M}$ admits manufactured dimension functions. The proof can be viewed as a technique for selecting "just the right" ultrafilter $\mathscr{U}$ for constructing $d^{\mathscr{U}}_{\mathrm{Hru}}$ or, alternatively, as selecting an infinite branch in a certain tree or a point in a certain Cantor space. Taking the latter point of view, in Section 4, we formulate a conjecture that if $Th(\mathcal{M})$ is rosy, then a generic point in this space induces a well-ordered manufactured dimension function.

1 Definitions and notation

We introduce almost all of the notation and necessary background material for our investigation of manufactured dimension functions. Subsection 1.1 accounts for just some basic notation that might not be completely standardized in the literature. In Subsection 1.2, we present the general definition of a dimension function as it is understood in this paper, and we review the construction of the Hrushovski dimension function (or fine dimension function) on an ultraproduct of finite structures. Next, in Subsection 1.3, we review definitions and basic results around amalgamation classes of finite structures and their generic models, and we present the definition of the Ramsey property in both its standard "finitary" form and in a form that is slightly more convenient in the context of the generic model of a Ramsey class. Finally, Subsection 1.4 is a review of the finite sub-model property and a compilation of some restatements (i.e. tameness) of the finite sub-model property for theories arise as generics of amalgamation classes.

1.1 Notation and conventions

Unless explicitly stated otherwise, the signature $\mathsf{sig}(\mathscr{L})$ of any language $\mathscr{L}$ in question consists of countably many relation symbols, finitely many constant symbols, and no function symbols. Infinite structures are denoted by calligraphic upper-case letters like $\mathcal{A}$ and $\mathcal{M}$ with universes A and M, respectively, and in general, our notation for such structures is more or less standard (see [20]). For finite structures, we use simple upper-case letters like A, B, C, and we identify finite structures with their universes. For a subset A of M, where $\mathcal{M}$ is an infinite structure, $\mathcal{M}[A]$ denotes the induced substructure of $\mathcal{M}$ with universe A (together with interpretations of constant symbols, if any), but we often write A instead of $\mathcal{M}[A]$ if no confusion is likely to arise.

A class of finite structures is usually denoted by $\mathbf{K}$ or some other bold upper-case letter, and these are always assumed to be closed under isomorphisms. We also assume that for every n, the set $\{A \in \mathbf{K}_\forall : |A| = n\} /\cong$ is finite, where $\mathbf{K}_\forall = \{A : A \leq B \in \mathbf{K}\}$.

If $\mathcal{M}$ is a structure and g is a permutation of M, then $g\mathcal{M}$ is the structure with universe $\mathcal{M}$ and interpretations $R^{g\mathcal{M}} = \left\{g\overline{a} : \overline{a} \in R^{\mathcal{M}}\right\}$, $c^{g\mathcal{M}} = g(c^{\mathcal{M}})$. If A is a finite structure with $A \subset M$ and g is a permutation of M, gA is defined similarly. If $\mathcal{M}$ is an $\aleph_0$-categorical structure, then we define

$$\mathrm{acl}[\mathcal{M}] = \{\mathrm{acl}(X) : X \subset_{\mathrm{fin}} M\}.$$

When $\mathcal{M}$ is the generic model of an amalgamation class $\mathbf{K}$ (see below), then $\mathbf{K}(\mathcal{M})$ is the set of induced substructures of $\mathcal{M}$ that are in $\mathbf{K}$.

For each n, $\mathsf{Def}^n(\mathcal{M})$ denotes the boolean algebra of definable subsets $X \subseteq M^n$, and $\mathsf{Def}(\mathcal{M}) = \bigcup_n \mathsf{Def}^n(\mathcal{M})$. For each n and any $S = \{i_0 < \cdots < i_{k-1}\} \subseteq n = \{0, 1, ..., n-1\}$, then $\pi_S : M^n \to M^k$ is the function given by $\pi_S(\overline{a}) = (a_{i_0}, ..., a_{i_{k-1}})$.

1.2 Dimension theory

We present a general definition of a dimension function on a structure $\mathcal{M}$, and then we present to the construction of the fine dimension, or Hrushovski dimension, on a structure obtained as an ultraproduct of finite structures.

Definition 1.1. A *dimension function* on a structure $\mathcal{M}$ is a map $\delta : \mathsf{Def}(\mathcal{M}) \to \mathbb{A} \cup \{-\infty\}$, where $\mathbb{A}$ is an ordered abelian group, satisfying the following:

D0. $\delta(\emptyset) = -\infty$.

D1. $\delta(X) = 0$ whenever X is finite and non-empty.

D2. $\delta(X \cup Y) = \max\{\delta(X), \delta(Y)\}$ for all $X, Y \in \mathsf{Def}(\mathcal{M})$ of the same arity.

D3. (Weak Fubini) For any $a \in \mathbb{A}$, any $X \in \mathsf{Def}(\mathcal{M})$ of arity n, and any $S \subseteq n$, if $\delta(\pi_S^{-1}(\overline{b})) \leq a$ for all $\overline{b} \in \pi_S X$, then $\delta(X) \leq \delta(\pi_S X) + a$

In practice, we will usually have $\mathbb{A} = \mathbb{R}$. At times, we will require a more exact form of the Fubini property, called the "$\mathcal{M}^+$-semi-exact-Fubini" (where $\mathcal{M}^+$ is a respectful expansion of $\mathcal{M}$) defined below – see Definition 2.1.

Let $G \leq Aut(\mathcal{M})$. We say that δ is G-invariant if for every $X \in \mathsf{Def}(\mathcal{M})$ and every $g \in G$, $\delta(gX) = \delta(X)$. If $G = Aut(\mathcal{M})$, then we just say that δ is invariant instead of $Aut(\mathcal{M})$-invariant. For $\aleph_0$-categorical countable $\mathcal{M}$, this is identical to what is called "continuity" in [12].

Definition 1.2 (Hrushovski dimension)**.** Let I be an infinite index set, and for each $i \in I$, let A_i be a finite $\mathscr{L}$-structure (fixing some language $\mathscr{L}$). Let $\mathscr{U}$ be a non-principal ultrafilter on I. Given these data, [12] describes a method for defining a dimension function $d^{\mathscr{U}}_{\mathrm{Hru}} : \mathsf{Def}(\mathcal{M}) \to \mathbb{R} \cup \{-\infty\}$ on $\mathcal{M}^* = \prod_i A_i/\mathscr{U}$.

- For $i \in I$ and a definable set X of $\mathcal{M}^*$ – say $X = \varphi(\mathcal{M}^*, \bar{b})$ where $\varphi(\bar{x}, \bar{y}) \in \mathscr{L}$, $m = |\bar{b}|$ and $\bar{b} = (b_0, ..., b_{m-1}) \in M^m$ – we define $X(A_i) = \varphi(A_i, f_0(i), ..., f_{m-1}(i))$ where $b_t = [f_t]_{\mathscr{U}}$ for each $t < m$.

- $\mathbb{R}^*$ be the ultrapower of $\mathbb{R} = (\mathbb{R}, +, \cdot, 0, 1, <, \exp, \log)$ along $\mathscr{U}$, and let $\mathfrak{C}_{\mathrm{fin}}$ be the convex hull of (standard) $\mathbb{Z}$ in $\mathbb{R}^*$. Then, of course, $\mathbb{R}^*$ is a real-closed field and so forth, and $\mathfrak{C}_{\mathrm{fin}}$ is a convex subring. Since we only care about order and additive structure, of more relevance is the fact that $\mathbb{R}^*$ is an ordered $\mathbb{R}$-vector space with $\mathfrak{C}_{\mathrm{fin}}$ as a subspace, so that $\mathbb{R}^*/\mathfrak{C}_{\mathrm{fin}}$ is, again, an ordered $\mathbb{R}$-vector space.

  Let $\pi : \mathbb{R}^* \to \mathbb{R}^*/\mathfrak{C}_{\mathrm{fin}}$ be the quotient map, and let $\mathbb{A}$ be the $\mathbb{R}$-subspace of $\mathbb{R}^*/\mathfrak{C}_{\mathrm{fin}}$ generated by $\{\pi(\log|M^*|)\}$, where $\log|M^*| = \left[i \mapsto \log|A_i|\right]_{\mathscr{U}}$. (Similarly, if X is a definable set, the $|X(\mathcal{M}^*)| := \left[i \mapsto \log|X(A_i)|\right]_{\mathscr{U}}$.) In fact, it is easy enough to check that the assignment $\pi(\log|M^*|) \mapsto 1$ induces an isomorphism of ordered $\mathbb{R}$-vector spaces $h : \mathbb{A} \to \mathbb{R}$, and it is convenient to simply identify $\mathbb{A}$ with $\mathbb{R}$ itself, but we will not do this quite yet.

- For the dimension function, we would initially define $d^{\mathscr{U}}_{\mathrm{Hru}} : \mathsf{Def}(\mathcal{M}^*) \to \mathbb{A} \cup \{-\infty\}$ by setting

  $$d^{\mathscr{U}}_{\mathrm{Hru}}(X) = \pi(\log|X(\mathcal{M}^*)|)$$

  for each non-empty $X \in \mathsf{Def}(\mathcal{M}^*)$, and $d^{\mathscr{U}}_{\mathrm{Hru}}(\emptyset) = -\infty$. To make this genuinely $\mathbb{R}$-valued, we would just reset $d^{\mathscr{U}}_{\mathrm{Hru}}(X) = h\left(\pi(\log|X(\mathcal{M}^*)|)\right)$, obviously.

- One observes that:

  1. For any $m, r > 0$, $h\Big(\pi\Big(\left[i \mapsto \log(m|A_i|^r)\right]_{\mathscr{U}}\Big)\Big) = r$.
  2. For $r \geq 0$ and a non-empty definable set X, $d^{\mathscr{U}}_{\mathrm{Hru}}(X) = r$ if and only if for every $\varepsilon > 0$, $\left\{i : \left|\frac{\log|X(A_i)|}{\log|A_i|} - r\right| < \varepsilon\right\} \in \mathscr{U}$.
  3. For definable sets X and Y, $d^{\mathscr{U}}_{\mathrm{Hru}}(X) = d^{\mathscr{U}}_{\mathrm{Hru}}(Y)$ iff there is a number $N \in \mathbb{N}$ such that $\left\{i : \frac{1}{N}|Y(A_i)| \leq |X(A_i)| \leq N|Y(A_i)|\right\} \in \mathscr{U}$.

Remark 1.3. In general, $d^{\mathscr{U}}_{\mathrm{Hru}}$ is not invariant or continuous in the sense [12]. To attain continuity, it is usually necessary to enrich the logic – either by adding cardinality-comparison quantifiers or dimension-comparison quantifiers, which are not first-order. Hence, this move to enforce invariance wanders far away from "traditional" model-theoretic domains like simple theories, so the following theorem of [6] is more or less optimal for relating $d^{\mathscr{U}}_{\mathrm{Hru}}$ to dividing and forking.

Theorem 1.4. *Let I, $\mathscr{U}$, $(A_i)_{i\in I}$, $\mathcal{M}$, and $d^{\mathscr{U}}_{\mathrm{Hru}}$ be as above. Let $X = \varphi(\mathcal{M}, \overline{a})$ and $Y = \psi(\mathcal{M}, \overline{b})$ such that $X \subseteq Y$. If $\varphi(\overline{x}, \overline{b})$ divides over $\overline{a}$, then there is some $\overline{b}' \equiv_{\overline{a}} \overline{b}$ such that $d^{\mathscr{U}}_{\mathrm{Hru}}(\varphi(\mathcal{M}, \overline{b}')) < d^{\mathscr{U}}_{\mathrm{Hru}}(\psi(\mathcal{M}, \overline{a}))$.*

In order to out-do this optimum, it seems, one must find a way to impose invariance using special properties of the underlying family $(A_i)_{i\in I}$ of finite structures. For example, if one knows *a priori* that $(A_i)_{i\in I}$ is drawn from an N-dimensional asymptotic class in the sense of [4], then $d^{\mathscr{U}}_{\mathrm{Hru}}$ is always continuous (because of the the definability clause in the formulation of asymptotic classes) and independent of the choice of $\mathscr{U}$. We have found, and so we demonstrate in Section 3, that the Ramsey property is an adequate special property for the existence of an invariant dimension theory that is "patterned on" $d^{\mathscr{U}}_{\mathrm{Hru}}$, although it is not identical to it.

1.3 Classes of finite structures

In this subsection, we defined several amalgamation-type properties that a class of finite structures might have. Conjoining several of these properties in different combinations yields the notions of Fraïssé classes and coherent classes which are the setting for structural Ramsey theory as discussed in this paper. Almost all of these properties are gathered together in Definition 1.5, and all of them have been remarked upon by other authors. A key fact for all of these conjunctions is the existence of generic models (sometimes called Fraïssé limits in certain cases)

Definition 1.5. Let $\mathbf{K}$ be a class of finite $\mathscr{L}$-structures. We list a number of properties that $\mathbf{K}$ might have.

1. (Joint-embedding property (JEP))

   For any $A, B \in \mathbf{K}$, there are $C \in \mathbf{K}$ and embeddings $f_A : A \to C$ and $f_B : B \to C$.

2. (Amalgamation property (AP))

   For any $A, B_1, B_2 \in \mathbf{K}$ and embeddings $f_i : A \to B_i$ $(i = 1, 2)$, there are $C \in \mathbf{K}$ and embeddings $f'_i : B_i \to C$ such that $f'_1 f_1 = f'_2 f_2$.

3. (Disjoint joint-embedding property (disjoint-JEP))

   For any $A, B \in \mathbf{K}$, there are $C \in \mathbf{K}$ and embeddings $f_A : A \to C$ and $f_B : B \to C$ such that $f_A A \cap f_B B = \emptyset$.

4. (Disjoint amalgamation property (disjoint-AP))

   For any $A, B_1, B_2 \in \mathbf{K}$ and embeddings $f_i : A \to B_i$ $(i = 1, 2)$, there are $C \in \mathbf{K}$ and embeddings $f'_i : B_i \to C$ such that $f'_1 f_1 = f'_2 f_2$ and $f'_1 B_1 \cap f'_2 B_2 = f'_1 f_1 A = f'_2 f_2 A$.

5. (Heredity property (HP))

   For every $B \in \mathbf{K}$, every induced substructure $A \leq B$ is in $\mathbf{K}$.

6. (Weak Löwenheim-Skolem property (WLSP))

   There is a function $\lambda : \mathbb{N} \to \mathbb{N}$ such that for any $A \in \mathbf{K}$ and any $X \subseteq A$, there are $A', B \in \mathbf{K}$ such that $A, A' \leq B$, $X \subseteq A'$, and $|A'| \leq \lambda(|X|)$.

We say that $\mathbf{K}$ is an *amalgamation class* if it has JEP and AP. As always, $\mathbf{K}$ is a *Fraïssé class* if it has JEP, AP, and HP. A class $\mathbf{K}$ is a *coherent class* if it has JEP, AP, and WLSP.

Theorem 1.6 (cf. Theorem 7.1.2 of [10])**.** *Let* $\mathbf{K}$ *be a class of finite structures. If* $\mathbf{K}$ *has both AP and JEP, then there is a countably infinite* generic model $\mathcal{M}$ *with the following three properties:*

- *(*$\mathbf{K}$*-universality) For every* $A \in \mathbf{K}$*, there is an embedding* $A \to \mathcal{M}$*.*
- *(*$\mathbf{K}$*-homogeneity) For any* $A, B \in \mathbf{K}$ *and any embedding* $f_0 : A \to \mathcal{M}$*, there is an embedding* $f : B \to \mathcal{M}$ *such that* $f_0 \subseteq f$*.*
- *(*$\mathbf{K}$*-closedness) For every* $X \subset_{fin} M$*, there are* $A \in \mathbf{K}$ *and an embedding* $f : A \to \mathcal{M}$ *such that* $X \subseteq fA$*.*

Any countable structure with these three properties is isomorphic to $\mathcal{M}$*; because of this uniqueness, we call* $\mathcal{M}$ the *generic model, and we see that the generic theory* $T_{\mathbf{K}} = Th(\mathcal{M})$ *is well-defined in terms of* $\mathbf{K}$*. Furthermore:*

- *[17]:* $T_{\mathbf{K}}$ *is* $\aleph_0$*-categorical and model-complete if and only if* $\mathbf{K}$ *has WLSP.*
- *[10]:* $T_{\mathbf{K}}$ *is* $\aleph_0$*-categorical and eliminates quantifiers if and only if* $\mathbf{K}_\forall$ *has AP and JEP (i.e.* $\mathbf{K}_\forall$ *is a Fraïssé class).*
- $\mathbf{K}$ *has disjoint-JEP if and only if* $\mathrm{acl}(\emptyset) = \emptyset$ *in* $\mathcal{M}$*.*
- $\mathbf{K}$ *has disjoint-AP if and only if for every embedding* $f : A \to \mathcal{M}$*, where* $A \in \mathbf{K}$*,* fA *is algebraically closed.*

For our purposes in this paper, coherent classes may be a little too general, so we will focus on the narrower concept of strongly coherent class. Directly from the definition, it is not difficult to see that a Fraïssé class with disjoint-JEP and disjoint-AP is necessarily a strongly coherent class.

Definition 1.7. Let $\mathbf{K}$ be a class of finite structures. We say that $\mathbf{K}$ is a *strongly coherent class* if it has disjoint-AP, WLSP, and if $\mathcal{M}$ is the generic model of $\mathbf{K}$, then $\mathrm{acl}[\mathcal{M}] \subseteq \mathbf{K}$. (The last condition is equivalent to requiring that $\mathbf{K}$ is the isomorphism-closure of $\mathrm{acl}[\mathcal{M}]$.) A strongly coherent class that is actually a Fraïssé class could be called a *strong Fraïssé class*.

Next, we formalize the Ramsey property as it pertains to strongly coherent classes. This is the classical, finitary formulation of this property; the so-called "amalgamated Ramsey property" described below is slightly more convenient for our purposes here in that it pertains to colorings of the generic model which arise fairly naturally.

Definition 1.8. Let $\mathbf{K}$ be a strongly coherent class.

- For $A \in \mathbf{K}$, we say that $\mathbf{K}$ has the A-Ramsey property if for every $B \in \mathbf{K}$ and every $k \in \mathbb{N}^+$, there is a $C \in \mathbf{K}$ such that $C \to (B)^A_k$, meaning that for every coloring $\xi : \mathrm{Emb}(A, C) \to \{0, ..., k-1\}$, there is an embedding $f : B \to C$ such that ξ is constant in $\mathrm{Emb}(A, fB)$.
- We say that $\mathbf{K}$ has the Ramsey property (and that $\mathbf{K}$ is a Ramsey class) if it has the A-Ramsey property for every $A \in \mathbf{K}$.

Fact 1.9 (See [9]). Let $\mathbf{K}$ be a strongly coherent class with generic model $\mathcal{M}$. The following are equivalent:

1. $\mathbf{K}$ is a Ramsey class.

2. $\mathcal{M}$ has the following "amalgamated Ramsey property":

   For every $A \in \mathbf{K}$ and every positive $k \in \mathbb{N}^+$, for any coloring $\xi : \mathrm{Emb}(A, \mathcal{M}) \to \{0, ..., k-1\}$, there is a system $(f_B)_{B \in \mathbf{K}(\mathcal{M})}$ of embeddings $f_B : B \to \mathcal{M}$ such that for each $B \in \mathbf{K}(\mathcal{M})$, ξ is constant on $\mathrm{Emb}(A, f_B B)$.

We will be principally interested in strongly coherent classes $\mathbf{K}$ such that $T_{\mathbf{K}}$ has the finite submodel property. The following theorem of [15], therefore, is a non-trivial obstacle for us, and (again following a definition from [15]) it suggests the notion of a respectful expansion below. When we define our notion of a manufactured dimension function, there will always be a respectful expansion in play.

Theorem 1.10. *Let $\mathbf{K}$ be a strongly coherent class with generic model $\mathcal{M}$. If $\mathbf{K}$ is a Ramsey class, then $\mathcal{M}$ carries a 0-definable linear ordering.*

Definition 1.11. Let $\mathbf{K}$ be a strongly coherent class with generic model $\mathcal{M}$, and let $\mathscr{L}$ be the language of $\mathbf{K}$ and $\mathcal{M}$.

- A *respectful expansion of* $\mathbf{K}$ is a strongly coherent class $\mathbf{K}^+$ in a language $\mathscr{L}^+$ satisfying the following:

  1. $\mathsf{sig}(\mathscr{L}^+)$ extends $\mathsf{sig}(\mathscr{L})$ by finitely many new relation symbols.
  2. For every $A \in \mathbf{K}^+$, $A{\restriction}\mathscr{L} \in \mathbf{K}$, and for every $A \in \mathbf{K}$, there is an $A^+ \in \mathbf{K}^+$ such that $A^+{\restriction}\mathscr{L} = A$.
  3. For every $A \in \mathbf{K}^+$, every $B \in \mathbf{K}$, and every embedding $f : A{\restriction}\mathscr{L} \to B$, there is a $B^+ \in \mathbf{K}^+$ such that $B^+{\restriction}\mathscr{L} = B$ and f is an embedding of $\mathscr{L}^+$-structures $A \to B^+$.

  We observe that if $\mathbf{K}^+$ is a respectful expansion of $\mathbf{K}$ with generic model $\mathcal{M}^+$, then $\mathcal{M}^+{\restriction}\mathscr{L} \models T_{\mathbf{K}}$, so we also say that $\mathcal{M}^+$ is a respectful expansion of $\mathcal{M}$.

- We say that $\mathbf{K}$ *is Ramsey-expandable* if it admits a respectful expansion $\mathbf{K}^+$ that has the Ramsey property.

In general, if $\mathcal{M}^+$ is the generic model of a respectful expansion of $\mathbf{K}^+$ of $\mathbf{K}$ (whose generic model is $\mathcal{M} = \mathcal{M}^+{\restriction}\mathscr{L}$), then $\mathcal{M}^+$ is "very close" to $\mathcal{M}$. This kind of closeness is indicated in the following lemma, which will also be useful in our analysis of manufactured dimension functions below.

Lemma 1.12. *Let $\mathbf{K}$ be a strongly coherent class with generic model $\mathcal{M}$, and let $\mathbf{K}^+$ be a respectful expansion of $\mathbf{K}$ with generic model $\mathcal{M}^+$ such that $\mathcal{M}^+{\restriction}\mathscr{L} = \mathcal{M}$. Let $A \subset_{fin} M$, and let $(\bar{b}_i)_{i<\omega}$ be an infinite A-$\mathscr{L}$-indiscernible sequence of n-tuples such that $\bar{b}_0 \cap \mathrm{acl}(A) = \emptyset$. Let $p(\bar{x}) \in S_n^{\mathcal{M}^+}(A)$ be consistent with $\mathrm{tp}^{\mathcal{M}}(\bar{b}/A)$. Then there is an A-$\mathscr{L}$-indiscernible sequence $(\bar{b}'_i)_{i<\omega}$ such that:*

- *$\mathcal{M}^+ \models p(\bar{b}'_i)$ for all $i < \omega$.*
- *For all $0 < k < \omega$, $\mathrm{tp}^{\mathcal{M}}(\bar{b}'_0 ... \bar{b}'_{k-1}) = \mathrm{tp}^{\mathcal{M}}(\bar{b}_0 ... \bar{b}_{k-1})$.*

1.4 Finite sub-model property, super-robustness, and tameness

In general, a structure $\mathcal{M}$ has the finite sub-model property if for every sentence $\varphi \in Th(\mathcal{M})$, for every $X \subset_{\text{fin}} M$, there is a finite induced substructure B of $\mathcal{M}$ such that $X \subseteq B$ and $B \vDash \varphi$.

In this paper, we are really only interested in $\aleph_0$-categorical structures, so we will refine the usual definition somewhat. So, for an $\aleph_0$-categorical structure $\mathcal{M}$, we say that $\mathcal{M}$ has the *finite sub-model property* (FSP) if, for every sentence $\varphi \in Th(\mathcal{M})$ and every $X \subset_{\text{fin}} M$, there is some $B \in \text{acl}[\mathcal{M}]$ such that $X \subseteq B$ and $B \vDash \varphi$. (Morally, this should be called something like the "algebraic finite sub-model property," but the extra words don't seem to be helpful in this paper.)

Definition 1.13. Let $\mathcal{M}$ be the generic model of a coherent class $\mathbf{K}$. We have already defined $\mathbf{K}(\mathcal{M})$. Extending this, for each $A \in \mathbf{K}(\mathcal{M})$, we set $\mathbf{K}_A(\mathcal{M}) = \{B \in \mathbf{K}(\mathcal{M}) : A \leq B\}$. It is not difficult to verify that

$$\mathscr{F}_0 = \{X \subseteq \mathbf{K}(\mathcal{M}) : (\exists A \in \mathbf{K}(\mathcal{M}))\, \mathbf{K}_A(\mathcal{M}) \subseteq X\}$$

is a proper filter on $\mathbf{K}(\mathcal{M})$.

Definition 1.14. Suppose $\mathcal{M}$ is the generic model of a strongly coherent class $\mathbf{K}$. For each $\varphi \in T_{\mathbf{K}}$, let

$$Y_\varphi = \{A \in \mathbf{K}(\mathcal{M}) : A \vDash \varphi\}.$$

Then, to say that $\mathcal{M}$ has FSP is precisely to say that

$$\mathsf{Cone} = \{X \subseteq \mathbf{K}(\mathcal{M}) : (\exists A \in \mathbf{K}(\mathcal{M}),\, \varphi \in T_{\mathbf{K}})\, \mathbf{K}_A(\mathcal{M}) \cap Y_\varphi \subseteq X\}$$

is a proper filter on $\mathbf{K}(\mathcal{M})$. Obviously, $\mathscr{F}_0 \subseteq \mathsf{Cone}$. We define $S_{\mathsf{Cone}}(\mathcal{M})$ to be the set of all ultrafilters $\mathscr{U}$ in $\mathbf{K}(\mathcal{M})$ such that $\mathsf{Cone} \subseteq \mathscr{U}$, and any $\mathscr{U} \in S_{\mathsf{Cone}}(\mathcal{M})$ is called a Cone-*ultrafilter*. Thus, FSP amounts to "$S_{\mathsf{Cone}}(\mathcal{M}) \neq \emptyset$."

Remark 1.15. Suppose $\mathcal{M}$ is the generic model of a strongly coherent class $\mathbf{K}$. If $\mathcal{M}$ has the finite sub-model property, then one can choose a function $f : T_{\mathbf{K}} \to \mathbb{N}$ such that the class

$$\hat{\mathbf{K}} = \{A \in \mathbf{K} : (\forall \varphi \in T_{\mathbf{K}})\, |A| \geq f(\varphi) \Rightarrow A \vDash \varphi\}$$

is also coherent with disjoint-JEP, disjoint-AP, and WLSP. Further, using the fact that $T_{\mathbf{K}}$ is model-complete, one finds that $\hat{\mathbf{K}}$ has the following additional property:

> *There is a function $\nu : \mathscr{L} \to \mathbb{N}$ such that for all $A, B \in \hat{\mathbf{K}}$, $\varphi(x_0, ..., x_{n-1}) \in \mathscr{L}$ and $\overline{a} \in A^n$, if $|A| \geq \nu(\varphi)$ and $A \leq B$, then $A \vDash \varphi(\overline{a}) \Leftrightarrow B \vDash \varphi(\overline{a})$.*

In [8] building on [19], this property is called *super-robustness* – that is, the sub-class $\hat{\mathbf{K}}$ itself is super-robust, while $\mathbf{K}$ (since $T_{\mathbf{K}} = T_{\hat{\mathbf{K}}}$ and $\hat{\mathbf{K}}$ is super-robust) is called *tame*. Briefly: a class is tame is if it has a co-final super-robust sub-class. We observe that if $\mathscr{F}_0$ were defined using $\hat{\mathbf{K}}$ in place of $\mathbf{K}$, then there is no need for the "second stage" in the definition of Cone.

Observation 1.16. Suppose $\mathcal{M}$ is the generic model of a strongly coherent tame class $\mathbf{K}$, and let $\mathscr{U} \in S_{\mathsf{Cone}}(\mathcal{M})$. Let $\mathcal{M}^* = \prod \mathbf{K}(\mathcal{M})/\mathscr{U} = \prod_{B \in \mathbf{K}(\mathcal{M})} B/\mathscr{U}$, and for each $a \in M$, let $j_a : \mathbf{K}(\mathcal{M}) \to M$ be some function such that $j_a(B) = a$ whenever $a \in B$. Then $f : \mathcal{M} \to \mathcal{M}^* : a \mapsto [j_a]_{\mathscr{U}}$ is an elementary embedding.

Remark 1.17. Since the generic model of a Ramsey class is definably linearly ordered, it is impossible for a Ramsey class to be tame. However, it is still possible for a tame class to be Ramsey-expandable. In fact, several well-known tame classes – such as the class of all finite graphs, the class of all affine spaces over $\mathbb{F}_q$ (q a fixed power of a prime), and several others – are Ramsey-expandable.

2 Manufactured dimension functions

We now introduce the main objects of study for the rest of this paper – manufactured dimension functions. We use the word "manufactured" here because some care is required in building them and ensuring invariance. As we will see, selecting the appropriate Cone-ultrafilter is a non-trivial hurdle, and this is in contradistinction to the general framework of pseudo-finite dimension functions, where any old ultrafilter will do and problems of definability/invariance are dealt with very blithely.

Definition 2.1 (Manufactured dimension functions)**.** Let $\mathcal{M}$ be the generic model of a strongly coherent tame class $\mathbf{K}$ with co-final super-robust sub-class $\hat{\mathbf{K}}$. Consider the following data:

- $\mathbf{K}^+$ is a respectful expansion of $\mathbf{K}$ with generic model $\mathcal{M}^+$ such that $\mathcal{M}^+\restriction\mathscr{L} = \mathcal{M}$.
- $\mathscr{V}$ is a non-principal ultrafilter on I.
- $\{C_i\}_{i\in I} \subseteq \hat{\mathbf{K}}(\mathcal{M})$ is a $\mathscr{V}$-co-final subset (meaning that for every $A \in \mathbf{K}(\mathcal{M})$, $A \leq C_i$ for $\mathscr{V}$-almost every i).
- $f : \mathcal{M} \to \mathcal{M}^* = \prod_i A_i/\mathscr{V}$ is an elementary embedding.

The data $(\mathbf{K}^+, \{C_i\}_i, \mathscr{V}, f)$ is called *admissible* if:

1. For all $\varphi(\overline{x}, \overline{y}) \in \mathscr{L}$ and all $\overline{b}, \overline{b}' \in M^{|\overline{y}|}$,
$$\mathrm{tp}^{\mathcal{M}^+}(\overline{b}) = \mathrm{tp}^{\mathcal{M}^+}(\overline{b}') \;\Rightarrow\; d^{\mathscr{V}}_{\mathrm{Hru}}(\varphi(\mathcal{M}^*, f\overline{b})) = d^{\mathscr{V}}_{\mathrm{Hru}}(\varphi(\mathcal{M}^*, f\overline{b}')).$$
2. The map $d^+ : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\} : \varphi(\mathcal{M}, \overline{b}) \mapsto d^{\mathscr{V}}_{\mathrm{Hru}}(\varphi(\mathcal{M}^*, f\overline{b}))$ has the following $\mathcal{M}^+$*-semi-exact-Fubini property*:

   > Suppose $A \in \mathbf{K}(\mathcal{M})$, X, Y are A-definable sets, and suppose $f : X \to Y$ is an A-definable surjective function. Let $\overline{b}_0, ..., \overline{b}_{k-1}$ be a complete set of representatives for the set of all $p \in S^{\mathcal{M}^+}_m(A)$ consistent with Y, where $m = \mathrm{ari}(Y)$. Then $d^+(X) = d^+(Y) + \max_{i<k} d^+(f^{-1}(\overline{b}_i))$.

Assuming $(\mathbf{K}^+, \{C_i\}_i, \mathscr{V}, f)$ is admissible, one may define a dimension function $d : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\}$ on $\mathcal{M}$ itself by setting

$$d(\varphi(\mathcal{M}, \overline{b})) = \max\left\{ d^{\mathscr{V}}_{\mathrm{Hru}}(\varphi(\mathcal{M}^*, f\overline{b}')) : \mathrm{tp}^{\mathcal{M}}(\overline{b}') = \mathrm{tp}^{\mathcal{M}}(\overline{b}) \right\}.$$

We call a dimension function d obtained in this manner a *manufactured dimension function.* We observe that a manufactured dimension function is necessarily $Aut(\mathcal{M})$-invariant.

Our two theorems regarding the behavior of manufactured dimension functions are the following, and their proofs take up the last two subsections of this section. The first theorem is a "dividing detection" result, and the second uses dividing detection to give a sufficient condition for super-simplicity of the generic theory.

Theorem 2.2 (cf. [6] and [7])**.** *Let $\mathbf{K}$ be a strongly coherent tame class with generic model $\mathcal{M}$, and let $d : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\}$ be a manufactured dimension function. Let $\psi(\overline{x}, \overline{a})$, $\varphi(\overline{x}, \overline{b}) \in \mathscr{L}(M)$ such that $\varphi(\mathcal{M}, \overline{b}) \subseteq \psi(\mathcal{M}, \overline{a})$. Then, if $\varphi(\overline{x}, \overline{b})$ divides over $\overline{a}$, then $d(\varphi(\mathcal{M}, \overline{b})) < d(\psi(\mathcal{M}, \overline{a}))$.*

Theorem 2.3 (cf. [5])**.** *Let $\mathbf{K}$ be a strongly coherent tame class with generic model $\mathcal{M}$, and let $d : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\}$ be a manufactured dimension function. If the set $\mathrm{img}\left(d\restriction\mathsf{Def}^1(\mathcal{M})\right)$ is well-ordered, then $T_{\mathbf{K}}$ is super-simple.*

2.1 Proof of Theorem 2.2

For all of this subsection, let $\mathbf{K}$ be a strongly coherent tame class with generic model $\mathcal{M}$, and let $d : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\}$ be a manufactured dimension function. Let $\hat{\mathbf{K}}$ be co-final super-robust subclass, and let $(\mathbf{K}^+, \{C_i\}_i, \mathscr{V}, f)$ be the admissible data underlying d, with $\mathcal{M}^* = \prod_i C_i/\mathscr{V}$ and $\mathcal{M}^+$ the generic model of $\mathbf{K}^+$ so that $\mathcal{M}^+\restriction\mathscr{L} = \mathcal{M}$. Let $\mathbb{R}^*$ be the ultrapower of $\mathbb{R}$ along $\mathscr{V}$. Altogether, we have a 3-sorted structure $(\mathcal{M}^+, \mathcal{M}^*, f, \mathbb{R}^*, (\kappa_\varphi)_\varphi)$, where for each partitioned formula $\varphi = \varphi(\overline{x}, \overline{y})$ of the language $\mathscr{L}$ of $\mathbf{K}$, $\kappa_\varphi : (M^*)^{|\overline{y}|} \to \mathbb{R}^*$ is given by $\kappa_\varphi(\overline{a}) = |\varphi(\mathcal{M}^*, \overline{a})|$.

Lemmas 2.5 and 2.6 below are components of the proof of Theorem 3.4 in [6], and their statements and proofs here are not significantly different from their analogs there except that one must spend a little more care obtaining enough saturation to carry them. This care amounts to the following lemma.

Lemma 2.4. *The 3-sorted structure $(\mathcal{M}^+, \mathcal{M}^*, f, \mathbb{R}^*, (\kappa_\varphi)_\varphi)$ has an elementary extension of the form $(\mathcal{N}, \mathcal{M}^*, f', \mathbb{R}^*, (\kappa_\varphi)_\varphi)$ such that $\mathcal{M}^+ \preceq \mathcal{N}$ which is $\aleph_0$-saturated with respect to subsets of $\mathcal{N}$.*

Proof. First, one passes to an $\aleph_0$-saturated (in the usual sense) extension of $(\mathcal{M}^+, \mathcal{M}^*, f, \mathbb{R}^*, (\kappa_\varphi)_\varphi)$. Next, one uses the fact that the 2-sorted structure $(\mathcal{M}^*, \mathbb{R}^*, (\kappa_\varphi)_\varphi)$ is $\aleph_1$-saturated to select $(\mathcal{N}, f')$ as specified. □

Lemma 2.5. *Let $(\mathcal{N}, \mathcal{M}^*, f', \mathbb{R}^*, (\kappa_\varphi)_\varphi)$ be as described in the previous lemma. Let $\varphi(\overline{x}, \overline{a}), \varphi_1(\overline{x}, \overline{b}) \in \mathscr{L}(M)$ such that $\varphi_1(\mathcal{N}, \overline{b}) \subseteq \varphi(\mathcal{N}, \overline{a})$. If $d^+(\varphi_1(\mathcal{N}, \overline{b})) = d^+(\varphi(\mathcal{N}, \overline{a}))$ whenever $\mathrm{tp}^{\mathcal{N}}(\overline{b}'/\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}/\overline{a})$, then there is a positive $C \in \mathbb{N}$ such that $C{\cdot}\big|\varphi_1(\mathcal{M}^*, f\overline{b}')\big| \geq \big|\varphi(\mathcal{M}^*, f\overline{a})\big|$ whenever $\mathrm{tp}^{\mathcal{N}}(\overline{b}'/\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}/\overline{a})$.*

Proof. Given $\overline{b}'$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{b}'/\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}/\overline{a})$, by the last of observation in Definition 1.2, there is a positive standard integer $n(\overline{b}')$ such that

$$\frac{1}{n(\overline{b}')}|X| \leq |\varphi_1(\mathcal{M}^*, f\overline{b}')| \leq n(\overline{b}')|X|$$

and from this, we derive $e^{n(\overline{b}')}{\cdot}|\varphi(\mathcal{M}^*, f\overline{b}')| \geq |X|$. We may assume that $n(\overline{b}')$ is chosen to be as small as possible. Towards a contradiction, suppose that for every $k \in \mathbb{N}$, there is some $\overline{b}'$ such that $\mathrm{tp}^{\mathcal{M}^+}(\overline{b}'/\overline{a}) = \mathrm{tp}^{\mathcal{M}^+}(\overline{b}/\overline{a})$ and $e^{n(\overline{b}')} > k$. It follows that

$$p(\overline{y}, \overline{a}) = \mathrm{tp}^{\mathcal{M}^+}(\overline{b}/\overline{a}) \cup \{k{\cdot}|\varphi(\mathcal{M}^*, \overline{y})| < |X| : \text{standard } k \in \mathbb{N}\}$$

is a consistent type over $\overline{a} \subset_{\mathrm{fin}} N$, so as $(\mathcal{N}, \mathcal{M}^*, f', \mathbb{R}^*, (\kappa_\varphi)_\varphi)$ is $\aleph_0$-saturated relative to subsets of $\mathcal{N}$, it is realized by some $\overline{c}$ in $\mathcal{N}$. Then, applying the same observation from Definition 1.2, we find that $\log|\varphi(\mathcal{M}^*, f'\overline{c})| < \log|X|$, so $d^+(\varphi_1(\mathcal{N}, \overline{c})) < d^+(X)$ – a contradiction. □

Lemma 2.6. *Let $(\mathcal{N}, \mathcal{M}^*, f', \mathbb{R}^*, (\kappa_\varphi)_\varphi)$ be as described in the previous lemma. Let $\varphi(\overline{x}, \overline{a}), \varphi_1(\overline{x}, \overline{b}) \in \mathscr{L}(M)$ such that $\varphi_1(\mathcal{N}, \overline{b}) \subseteq \varphi(\mathcal{N}, \overline{a})$ and $\overline{a} \subseteq \overline{b}$. Further, suppose that $\varphi_1(\overline{x}, \overline{b})$ divides over $\overline{a}$ by way of an $\overline{a}$-indiscernible sequence $(\overline{b}_i)_{i<\omega}$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{b}_i/\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}/\overline{a})$ for all $i < \omega$. Then $d^+(\varphi_1(\mathcal{N}, \overline{b})) < d^+(\varphi(\mathcal{N}, \overline{a}))$.*

For the proof of Lemma 2.6, we will also need the following clever measure-theoretic fact.

Proposition 2.7 (Proposition 3.3 of [6]). *Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $\varepsilon > 0$. Let $A_0, ..., A_i, ... \subseteq X$ be measurable sets such that $\mu(A_i) \geq \varepsilon$ for every i. Then, for every positive $k \in \mathbb{N}$, there are $i_0 < \cdots < i_{k-1}$ such that $\mu\left(A_{i_0} \cap \cdots \cap A_{i_{k-1}}\right) \geq \varepsilon^{3^k}$.*

Proof of Lemma 2.6. We assume that $\left\{\varphi_1(\overline{x}, \overline{b}_i)\right\}_{i<\omega}$ is k-inconsistent, where $k \geq 2$. Towards a contradiction, suppose $d^+(\varphi_1(\mathcal{N}, \overline{b}')) = d^+(\varphi(\mathcal{N}, \overline{a}))$ whenever $\mathrm{tp}^{\mathcal{N}}(\overline{b}'/\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}/\overline{a})$. By Lemma 2.5, there is a positive $C \in \mathbb{N}$ such that $C \cdot \left|\varphi_1(\mathcal{M}^*, f\overline{b}')\right| \geq \left|\varphi(\mathcal{M}^*, f\overline{a})\right|$ whenever $\mathrm{tp}^{\mathcal{N}}(\overline{b}'/\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}/\overline{a})$.

Let $\mathcal{B}_0$ be the boolean algebra of definable subsets of $X = \varphi(\mathcal{N}, \overline{a})$, and let $\mathcal{B}$ be the σ-algebra on X generated by $\mathcal{B}_0$. Let $\mu : \mathcal{B} \to [0, 1]$ be the Loeb measure induced by the pre-measure $\mu_0 : \mathcal{B}_0 \to [0, 1]$ given by $\mu_0(X_0) = \mathsf{st}\left(\frac{|X_0|}{|X|}\right)$. Then $\mu(\varphi_1(\mathcal{M}^*, f'\overline{b}_i)) \geq 1/C$ for every $i < \omega$, so by Proposition 2.7, there are $i_0 < \cdots < i_{k-1} < \omega$ such that

$$\mu\Big(\varphi_1(\mathcal{M}^*, f'\overline{b}_{i_0}) \cap \cdots \cap \varphi_1(\mathcal{M}^*, f'\overline{b}_{i_{k-1}})\Big) \geq C^{-3k} > 0.$$

This implies that $\left\{\varphi_1(\overline{x}, \overline{b}_i)\right\}_{i<\omega}$ is *not* k-inconsistent – a contradiction. Thus, there is some $\overline{b}'$ in $\mathcal{N}$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{b}'/\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}/\overline{a})$ and $d^+(\varphi_1(\mathcal{N}, \overline{b}')) < d^+(\varphi(\mathcal{N}, \overline{a}))$. By the $Aut(\mathcal{M}^+)$-invariance of d^+, it follows that $d^+(\varphi_1(\mathcal{N}, \overline{b})) < d^+(\varphi(\mathcal{N}, \overline{a}))$. □

Proof of Theorem 2.2. By $\aleph_0$-categoricity of $\mathcal{M}$ and requirement D2, we may assume that $\varphi(\overline{x}, \overline{b})$ isolates a complete type $q(\overline{x}, \overline{b}\overline{a})$ over $\overline{a}\overline{b}$, that $\psi(\overline{x}, \overline{a})$ isolates a complete type $p(\overline{x}, \overline{a})$ over $\overline{a}$, and that $q(\mathcal{M}, \overline{b}\overline{a}) \subseteq p(\mathcal{M}, \overline{a})$. Now, $q(\overline{x}, \overline{b}, \overline{a})$ divides over $\overline{a}$, so we may also assume that $\overline{b} \cap \mathrm{acl}(\overline{a}) = \emptyset$. Towards a contradiction, suppose $d(q(\mathcal{M}, \overline{b}\overline{a})) = d(p(\mathcal{M}, \overline{a}))$.

Let $\overline{b}'\overline{a}' \equiv \overline{b}\overline{a}$ such that $d^+(q(\mathcal{M}, \overline{b}'\overline{a}')) = d(q(\mathcal{M}, \overline{b}\overline{a}))$. Applying the $\mathcal{M}^+$-semi-exaxct-Fubini property, we find that $d^+(p(\mathcal{M}, \overline{a}')) = d(p(\mathcal{M}, \overline{a}))$. Let $(\overline{b}_i)_{i<\omega}$ be an $\overline{a}$-$\mathscr{L}$-indiscernible sequence such that $\left\{q(\overline{x}, \overline{b}_i, \overline{a})\right\}_{i<\omega}$ is k-inconsistent for some k. If $g \in Aut(\mathcal{M})$ takes $\overline{a}\overline{b}$ to $\overline{a}'\overline{b}'$, then $(\overline{b}_i')_{i<\omega}$ is $\overline{a}'$-$\mathscr{L}$-indiscernible (where $\overline{b}_i' = g\overline{b}_i$) and $\left\{q(\overline{x}, \overline{b}_i', \overline{a}')\right\}_{i<\omega}$ is also k-inconsistent. By Lemma 1.12, we may also assume that $\mathrm{tp}^{\mathcal{M}^+}(\overline{b}_i'/\overline{a}') = \mathrm{tp}^{\mathcal{M}^+}(\overline{b}'/\overline{a}')$ for all $i < \omega$, and then by Lemma 2.6, we find that

$$d(q(\mathcal{M}, \overline{b}\overline{a})) = d^+(q(\mathcal{M}, \overline{b}'\overline{a}')) < d^+(p(\mathcal{M}, \overline{a}')) = d(p(\mathcal{M}, \overline{a}))$$

as desired. □

2.2 Proof of Theorem 2.3

Again, let $\mathbf{K}$ be a strongly coherent tame class with generic model $\mathcal{M}$, and let $d : \mathsf{Def}(\mathcal{M}) \to \mathbb{R} \cup \{-\infty\}$ be a manufactured dimension function. Also, let $\hat{\mathbf{K}}$ be co-final super-robust subclass, and let $(\mathbf{K}^+, \{C_i\}_i, \mathscr{V}, f)$ be the admissible data underlying d.

Definition 2.8. For each $n = 1, 2, ...$, let $R_n(d) = \{d(X) : \text{non-empty } X \in \mathsf{Def}^n(\mathcal{M})\}$, and let $R_1^+(d)$ be the additive sub-semi-group of $(\mathbb{R}_{\geq 0}, +)$ generated by the subset $R_1(d)$.

Fact 2.9. Let $X \subseteq \mathbb{R}_{\geq 0}$, and define $\Sigma^0 X = \{0\}$ and $\Sigma^{n+1} X = \Sigma^n X + X = \{v + x : v \in \Sigma^n X, x \in X\}$ for each $n \in \mathbb{N}$. If X is well-ordered, then every $\Sigma^n X$ is also well-ordered.

Lemma 2.10 (cf. Theorem 3.4 of [3]). $\mathrm{img}(d) \subseteq R_1^+(d) \cup \{-\infty\}$. *In fact, $R_n(d) \subseteq \Sigma^n R_1(d)$ for every n.*

Proof. By induction on n, we show that $R_n(d) \subseteq R_1^+(d)$ for every $n = 1, 2, ...$ It's obvious that $R_1(d) \subseteq R_1^+(d)$, so inductively, we assume that $R_k(d) \subseteq \Sigma^k R_1(d)$ for each $k = 1, 2, ..., n$, and we prove that $R_{n+1}(d) \subseteq \Sigma^{n+1} R_1(d)$.

So, let $\varphi(w, x_0, ..., x_{n-1}, y_0, ..., y_{m-1}) \in \mathscr{L}$, and let $\overline{a} \in M^m$. We define a partitioned formula $\varphi_1(w; \overline{x}, \overline{y}) = \varphi(w, \overline{x}, \overline{y})$. Let $\overline{b}_0, ..., \overline{b}_{k-1}$ be a complete set of representatives for the set of all $p \in S_m^{\mathcal{M}^+}(\overline{a})$ consistent with the formula $\psi(\overline{y}, \overline{a}) = \exists w\, \varphi(w, \overline{x}, \overline{a})$. Then by the $\mathcal{M}^+$-semi-exact-Fubini property, we have

$$\begin{aligned} d^+(\varphi(\mathcal{M}, \overline{a})) &= d^+(\psi(\mathcal{M}, \overline{a})) + \max_{i<k} d^+(\varphi_1(\mathcal{M}; \overline{b}_i, \overline{a})) \\ &\in R_n(d) + R_1(d) \\ &\subseteq \Sigma^n R_1(d) + R_1(d) \\ &= \Sigma^{n+1} R_1(d) \\ &\subseteq R_1^+(d) \end{aligned}$$

which suffices to complete the proof of the lemma. □

Proof of Theorem 2.3. We are assuming now that $R_1(d)$ is well-ordered and that d has the restriction property. From the former, we know that for each n, the foundation rank function rk_n on $R_n(d)$ is ordinal-valued. By Theorem 2.2 and requirement D2, we know that for all $A, B \in \mathbf{K}(\mathcal{M})$ with $A \leq B$, for all n and $q(\overline{x}) \in S_n^{\mathcal{M}}(B)$, if $q(\overline{x})$ forks over A, then $d(q(\mathcal{M})) < d(p(\mathcal{M}))$ where $p = q{\restriction}A$. Thus, for any complete n-type p over some $A \in \mathbf{K}(\mathcal{M})$, $D(p) < \mathrm{rk}_n\big(d(p)\big) < \omega$. Since every definable set of $\mathcal{M}$ is definable by a disjunction over $\bigcup_{A \in \mathbf{K}(\mathcal{M})} S^{\mathcal{M}}(A)$, it follows that $D(X) < \infty$ for every definable set X of $\mathcal{M}$ – so $T_{\mathbf{K}}$ is super-simple. □

3 Existence of an invariant dimension function

We have verified that manufactured dimension functions detect dividing and, given well-ordering, even detect super-simplicity. However, we have not yet verified that any structures actually have manufactured dimension functions. In this section, we remedy this situation, proving the following theorem.

Theorem 3.1. *Let $\mathbf{K}$ be a strongly coherent tame class with a respectful expansion $\mathbf{K}^+$ that has the Ramsey property, and let $\mathcal{M}$ be the generic model of $\mathbf{K}$. Then $\mathcal{M}$ admits a manufactured dimension function.*

For all of this section, we fix a strongly coherent tame class $\mathbf{K}$ with $\hat{\mathbf{K}}$ as a co-final super-robust subclass. We also fix a respectful expansion $\mathbf{K}^+$ of $\mathbf{K}$ that has the Ramsey property. $\mathcal{M}^+$ is the generic model of $\mathbf{K}^+$, and $\mathcal{M} = \mathcal{M}^+{\restriction}\mathscr{L}$ is the generic model of $\mathbf{K}$.

3.1 Families of parameters

In the next subsection, we will return the setting of asymptotics of definable sets, but first, we will discuss a more general phenomenon in Ramsey-expandable tame class. Speaking very roughly, we take an obverse view of the Ramsey property – ordinarily, we think of colorings of copies of A inside larger structures $B, C \in \mathbf{K}^+$, but now we will think about finite-valued parameters defined on A-labeled members of $\hat{\mathbf{K}}(\mathcal{M})$ (in the terminology of [18]). Exactly what we mean by a parameter is exposed in the following definition.

Definition 3.2. For $A \in \mathbf{K}(\mathcal{M})$ and $B \in \hat{\mathbf{K}}_A(\mathcal{M})$, let (B, A) denote the expansion of B with new constants for the elements of A. If $A \in \mathbf{K}(\mathcal{M})$, $B \in \hat{\mathbf{K}}_A(\mathcal{M})$, and $B' \in \hat{\mathbf{K}}_{gA}(\mathcal{M})$ for some $g \in Aut(\mathcal{M})$, then $(B, A) \cong (B', gA)$ means that $g{\restriction}A$ extends to an isomorphism $B \to B'$. If $A \in \mathbf{K}(\mathcal{M})$, then $\hat{\mathbf{K}}^A(\mathcal{M})$ denotes the set of structures (B, gA), where $g \in Aut(\mathcal{M})$ and $B \in \hat{\mathbf{K}}_{gA}(\mathcal{M})$, with notion of isomorphism defined above.

Now, an *A-parameter* is just an isomorphism invariant function $f : \hat{\mathbf{K}}^A(\mathcal{M}) \to \{0, ..., k-1\}$ for some $k \in \mathbb{N}^+$. More generally, a *parameter* f is an A-parameter for some A, and when we do not initially specify this underlying A, we understand that f is an A_f-parameter with range $\{0, ..., k_f - 1\}$ – i.e. $f : \hat{\mathbf{K}}^{A_f}(\mathcal{M}) \to \{0, ..., k_f - 1\}$ is isomorphism invariant.

Around a family of parameters, we are interested in uniformizing all of them simultaneously on "large" sets of A-labeled structures in sense of Theorem 3.4 below. The notation required for Theorem 3.4 is first given in the next definition.

Definition 3.3. If f is an parameter, $g \in Aut(\mathcal{M}^+)$, and $i < k_f$, then

$$X(f, g; i) = \left\{B \in \hat{\mathbf{K}}(\mathcal{M}) : f(B, gA_f) = i\right\}$$

Now, let F be a finite set of parameters, and let $G_0 \subset_{\text{fin}} Aut(\mathcal{M}^+)$. Then

$$Z(F, G_0) = \bigcup_{\ell \in \prod_{f \in F}\{0,...,k_f-1\}} Z(F, G_0; \ell)$$

where for each $\ell \in \prod_{f\in F}\{0, ..., k_f - 1\}$,

$$Z(F, G_0; \ell) = \bigcap_{f \in F} \bigcap_{g \in G_0} X(f, g; \ell_f).$$

Theorem 3.4. *Let F be a countable set of parameters. Then there are a selection $\sigma \in \prod_{f\in F}\{0, ..., k_f - 1\}$ and an ultrafilter $\mathscr{U} \in S_{\mathsf{Cone}}(\mathcal{M})$ such that for all finite $F_0 \subseteq_{fin} F$ and $G_0 \subset_{fin} Aut(\mathcal{M}^+)$, $Z(F_0, G_0; \sigma{\restriction}F_0) \in \mathscr{U}$.*

To prove the theorem, we need to find/construct an ultrafilter, and as usual, this amounts to proving that the family of sets in question has the finite intersection property. This part of the plan is carried in the following lemma.

Lemma 3.5. *Let F be a finite set of parameters, and let $G_0 \subset_{fin} Aut(\mathcal{M}^+)$. Then, for every $A \in \mathbf{K}(\mathcal{M})$, $Z(F, G_0) \cap \hat{\mathbf{K}}_A(\mathcal{M}) \neq \emptyset$.*

Proof. Let $\mathscr{V}$ be an arbitrary Cone-ultrafilter, and let $B = \operatorname{acl}\left(A \cup \bigcup_{f\in F} \bigcup_{g\in G_0} gA_f\right)$. For each $f \in F$, we define a coloring $\xi_f : \operatorname{Emb}(A_f, \mathcal{M}^+) \to \{0, ..., k_f - 1\}$ by setting

$$\xi_f(e) = i \Leftrightarrow \{C \in \mathbf{K}(\mathcal{M}) : f(C, eA_f) = i\} \in \mathscr{V}$$

for each embedding $e : A_f \to \mathcal{M}^+$ (where we identify A_f with the induced substructure $\mathcal{M}^+[A_f]$ on the same set). By the amalgamated Ramsey property (applied $|F|$-many times), there is an automorphism $h \in Aut(\mathcal{M}^+)$ such that each ξ_f on $\mathrm{Emb}(A_f, hB)$. For each $f \in F$, let $\ell_f = \xi_f(hu)$ for some/any $u \in \mathrm{Emb}(A_f, B)$, and then let

$$X = \left\{ C \in \mathbf{K}(\mathcal{M}) : \bigwedge_{f \in F} (\forall u \in \mathrm{Emb}(A_f, B))\, f(C, huA_f) = \ell_f \right\}.$$

Then, $h^{-1}X \subseteq Z(F, G_0) \cap \hat{\mathbf{K}}_A(\mathcal{M})$, and since $X \in \mathscr{V}$, this suffices. □

Proof of Theorem 3.4. Without loss of generality, let $f_0, ..., f_n, ...$ be an enumeration of F, and for each n, let $F_n = \{f_0, ..., f_{n-1}\}$. Then, the following observation is immediate from Lemma 3.5:

Observation. For each n, there is a selection $\ell \in \prod_{i<n}\{0, ..., k_{f_i} - 1\}$ such that for all $G_0 \subset_{\mathrm{fin}} Aut(\mathcal{M}^+)$, for all $A \in \mathbf{K}(\mathcal{M})$, $Z(F_n, G_0; \ell) \cap \hat{\mathbf{K}}_A(\mathcal{M}) \neq \emptyset$.

Now, we set $P = \bigcup_n P_n$, where for each n, $P_n =$

$$\left\{ \ell \in \prod_{i<n}\{0, ..., k_{f_i} - 1\} : (\forall G_0 \subset_{\mathrm{fin}} Aut(\mathcal{M}^+))(\forall A \in \mathbf{K}(\mathcal{M}))\, Z(F_n, G_0; \ell) \cap \hat{\mathbf{K}}_A(\mathcal{M}) \neq \emptyset \right\}$$

and we view P as a tree under containment/initial segment. It is obviously finitely-branching, and by Lemma 3.5 again, it is infinite. Thus, by König's Lemma, P has an infinite branch – which amounts to a selection $\sigma \in \prod_{f \in F}\{0, ..., k_f - 1\}$, meaning that the family

$$\{X \subseteq \mathbf{K}(\mathcal{M}) : (\exists n, G_0, A)\, Z(F_0, G_0; \sigma{\restriction}F_0) \cap \mathbf{K}_A(\mathcal{M}) \subseteq X\}$$

has the finite intersection property. Since it has the finite intersection property, it extends to an ultrafilter $\mathscr{U} \in S_{\mathsf{Cone}}(\mathcal{M})$ – as desired. □

3.2 $Aut(\mathcal{M}^+)$-invariant dimension functions

In this subsection, we apply Theorem 3.4 to recovering manufactured dimension functions on the generic model $\mathcal{M}$ of our Ramsey-expandable tame class $\mathbf{K}$. Specifically, Theorem 3.4 is needed for imposing $Aut(\mathcal{M}^+)$-invariance. For this application, we first isolate a sufficiently-representative family of definable sets in the following definition, and we turn to the construction immediately thereafter – defining an appropriate countably infinite family of parameters and then applying Theorem 3.4 to find the ultrafilter for a tuple of admissible data.

Definition 3.6. We define $\mathcal{D}_n$ to be the set of existential $\mathscr{L}(M)$-formulas $\varphi(\overline{x}, \overline{a})$ such that:

- $\overline{x} = (x_0, ..., x_{n-1})$ is non-repeating.
- $\overline{a}$ enumerates some $A \in \mathbf{K}(\mathcal{M})$.
- $\varphi(\overline{x}, \overline{a})$ is a finite disjunction of complete types $\mathrm{tp}^{\mathcal{M}}(\overline{b}/A)$.

We then define $\mathcal{D} = \bigcup_n \mathcal{D}_n$. We take it as clear that every $X \in \mathsf{Def}(\mathcal{M})$ is definable by some $\varphi \in \mathcal{D}$.

Construction. For each n, each $\varphi = \varphi(\overline{x}, \overline{a}) \in \mathcal{D}_n$ where $\overline{a}$ enumerates $A \in \mathbf{K}(\mathcal{M})$, and each $r \in \mathbb{N}^+$, we define an A-parameter $f_{\varphi,r} : \hat{\mathbf{K}}^A(\mathcal{M}) \to \{0, 1, ..., n{\cdot}2^r - 1\}$ by setting

$$f_{\varphi,r}(B, gA) = \min\left\{t < n{\cdot}2^r : t{\cdot}2^{-r} \leq \frac{\log|\varphi(B, g\overline{a})|}{\log|B|} \leq (t+1){\cdot}2^{-r}\right\}.$$

Finally, we define $F_{\dim} = \{f_{\varphi,r} : \varphi \in \mathcal{D},\, r \in \mathbb{N}^+\}$, which is obviously a countable set of parameters. By Theorem 3.4, there are a selection $\sigma \in \prod_{f \in F_{\dim}}\{0, ..., k_f - 1\}$ and an ultrafilter $\mathscr{U} \in S_{\mathsf{Cone}}(\mathcal{M})$ such that for all finite $F_0 \subseteq_{\text{fin}} F_{\dim}$ and $G_0 \subset_{\text{fin}} Aut(\mathcal{M}^+)$, $Z(F_0, G_0; \sigma{\restriction}F_0) \in \mathscr{U}$.

As an aside, we define a function $d^+_\sigma : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\}$ by setting $d^+_{\mathscr{U}}(\emptyset) = -\infty$ and for consistent $\varphi(\overline{x}, \overline{a}) \in \mathcal{D}_n$,

$$d^+_\sigma(\varphi(\mathcal{M}, \overline{a})) = c \;\Leftrightarrow\; \bigcap_{r=1}^{\infty}\left[\sigma_{f_{\varphi,r}}{\cdot}2^{-r}, (\sigma_{f_{\varphi,r}} + 1){\cdot}2^{-r}\right] = \{c\}$$

We note that d^+_σ is well-defined because σ has a compatible ultrafilter $\mathscr{U}$, but $\mathscr{U}$ is not actually used in the specification d^+_σ.

However, using $\mathscr{U}$ we may also define $\mathcal{M}^* = \prod \mathbf{K}(\mathcal{M})/\mathscr{U} = \prod_{B \in \mathbf{K}(\mathcal{M})} B/\mathscr{U}$ and an elementary embedding $f : \mathcal{M} \to \mathcal{M}^* : a \mapsto [j_a]_{\mathscr{U}}$, where for each $a \in M$, $j_a : \mathbf{K}(\mathcal{M}) \to M$ is some function such that $j_a(B) = a$ whenever $a \in B \in \mathbf{K}(\mathcal{M})$. (This map is elementary because $\hat{\mathbf{K}}$ is super-robust and $\hat{\mathbf{K}}(\mathcal{M}) \in \mathscr{U}$; alternatively, because of the finite sub-model property and model-completeness of $T_{\mathbf{K}}$.) Setting $\hat{\mathscr{U}} = \left\{X \cap \hat{\mathbf{K}}(\mathcal{M}) : X \in \mathscr{U}\right\}$ (which is a Cone-ultrafilter for $\hat{\mathbf{K}}(\mathcal{M})$), we have collected data $(\mathbf{K}^+, \hat{\mathbf{K}}(\mathcal{M}), \hat{\mathscr{U}}, f)$. Although we have not yet shown that these data are admissible, we may still define a function $d^+ : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\}$ by setting $d^+(X) = d^{\hat{\mathscr{U}}}_{\text{Hru}}(\varphi(\mathcal{M}^*, f\overline{a}))$ whenever the formula $\varphi(\overline{x}, \overline{a}) \in \mathcal{D}$ defines non-empty $X \in \mathsf{Def}(\mathcal{M})$.

It will also be convenient to define $\mathcal{N} = \prod_{B \in \hat{\mathbf{K}}(\mathcal{M})} \mathcal{M}^+[B]/\hat{\mathscr{U}}$, while noting that it is not a model of $T_{\mathbf{K}^+}$. However, it is not difficult to see that for every $A \subset_{\text{fin}} N$, there is an embedding $\mathcal{N}[A] \to \mathcal{M}^+$.

The demonstration that $(\mathbf{K}^+, \hat{\mathbf{K}}(\mathcal{M}), \hat{\mathscr{U}}, f)$ is admissible culminates in Proposition 3.11, but its proof amounts to Lemmas 3.7 and 3.8 and Observation 3.9. The proof of Theorem 3.1 is completed by Proposition 3.11.

Lemma 3.7. *d^+ is well-defined, and in fact, $d^+ = d^+_\sigma$.*

Proof. Well-definedness follows from the finite sub-model property. That is to say, if X is defined by $\psi(\overline{x}, \overline{b}) \in \mathcal{D}$ and also by $\varphi(\overline{x}, \overline{a}) \in \mathcal{D}$ – so that $\mathcal{M} \models \forall\overline{x}\left(\psi(\overline{x}, \overline{b}) \leftrightarrow \varphi(\overline{x}, \overline{a})\right)$ – then we have $\left\{B \in \hat{\mathbf{K}}(\mathcal{M}) : B \models \forall\overline{x}\left(\psi(\overline{x}, \overline{b}) \leftrightarrow \varphi(\overline{x}, \overline{a})\right)\right\} \in \hat{\mathscr{U}}$, and it follows that for every r, the set $\left\{B : f_{\psi,r}(B, \overline{b}) = f_{\varphi,r}(B, \overline{a})\right\}$ is in $\hat{\mathscr{U}}$ as well. (The last statement is immediate from the definitions of d^+ and d^+_σ.) □

Lemma 3.8. *Let $\varphi(\overline{x}, \overline{a}) \in \mathscr{L}$, so that $\varphi(\overline{x}, \overline{y})$ is an existential $\mathscr{L}$-formula. Let $\overline{b}, \overline{b}' \in M^*$ such that $\text{tp}^{\mathcal{M}^*}(\overline{b}) = \text{tp}^{\mathcal{M}^*}(\overline{b}') = \text{tp}^{\mathcal{M}}(\overline{a})$, and let $u : \mathcal{N}[\overline{b}] \to \mathcal{M}^+$ and $u' : \mathcal{N}[\overline{b}'] \to \mathcal{M}^+$ be embeddings. Then, if $\text{tp}^{\mathcal{M}^+}(u\overline{b}) = \text{tp}^{\mathcal{M}^+}(u'\overline{b}')$, then $d^{\hat{\mathscr{U}}}_{\text{Hru}}(\varphi(\mathcal{M}^*, \overline{b})) = d^{\hat{\mathscr{U}}}(\varphi(\mathcal{M}^*, \overline{b}'))$.*

It follows that d^+ is $Aut(\mathcal{M}^+)$-invariant, and more generally, for every $\varphi(\overline{x}, \overline{a}) \in \mathcal{D}$, there are finitely many $d_0, ..., d_{m-1} \in \mathbb{R}\cup\{-\infty\}$ and a family $\theta_0(\overline{y}), ..., \theta_{m-1}(\overline{y})$ of quantifier-free $\mathscr{L}^+$-formulas such that for all $\overline{b}$ in $\mathcal{M}^*$, $d^{\hat{\mathscr{U}}}_{\text{Hru}}(\varphi(\mathcal{M}^*, \overline{b})) \in \{d_0, ..., d_{m-1}\}$ and $d^{\hat{\mathscr{U}}}_{\text{Hru}}(\varphi(\mathcal{M}^*, \overline{b})) = d_i \Leftrightarrow \mathcal{N} \models \theta_i(\overline{b})$.

Proof. Suppose $\varphi(\overline{x},\overline{a}) \in \mathcal{D}$ and $g \in Aut(\mathcal{M}^+)$ such that $u'\overline{b}' = gu\overline{b}$. Then for every r, the set $\left\{B : f_{\varphi,r}(B,u\overline{b}) = f_{\varphi,r}(B,gu\overline{b}')\right\}$ is in $\hat{\mathscr{U}}$ by construction, so

$$d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(\varphi(\mathcal{M}^*,u\overline{b})) = d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(\varphi(\mathcal{M}^*,u'\overline{b}')).$$

Since $S_n(T_{\mathbf{K}^+})$ is finite for every n, we recover $d_0,...,d_{m-1}$ and $\theta_0,...,\theta_{m-1}$ as described. That these θ_i's are quantifier-free comes from the fact that $\mathcal{N}[\overline{b}] \in \mathbf{K}^+$. $\square$

Observation 3.9. Suppose $X = \varphi_0(\mathcal{M},\overline{a}_0)$, $Y = \varphi_1(\mathcal{M},\overline{a}_1)$ where $\varphi_0,\varphi_1 \in \mathcal{D}_n$.

- If X is finite and non-empty, then $d^+(X) = 0$.
- $d^+(X \cup Y) = \max\{d^+(X), d^+(Y)\}$.

Lemma 3.10. *d^+ has the $\mathcal{M}^+$-semi-exact-Fubini property.*

More precisely, let $\theta_0(\overline{x},\overline{a}) \in \mathcal{D}_{n_0}$, $\theta_1(\overline{y},\overline{a}) \in \mathcal{D}_{n_1}$, $\varphi(\overline{x},\overline{y},\overline{a}) \in \mathcal{D}_{n_0+n_1}$ such that φ defines a surjective function $f : \theta_0(\mathcal{M},\overline{a}) \to \theta_1(\mathcal{M},\overline{a})$. Further, assume that $\theta_0(\overline{x},\overline{a})$ and $\theta_1(\overline{y},\overline{a})$ are both equivalent to complete types $p_0(\overline{x},\overline{a}_0)$ and $p_1(\overline{y},\overline{a}_0)$ of $\mathcal{M}$ over some $\overline{a}_0 \subset \overline{a}$ and that there are $\overline{b}_0,...,\overline{b}_{k-1} \in \|\overline{a}\|^{n_1}$ such that:

- *For each $i < k$, $\mathcal{M} \vDash \theta_1(\overline{b}_i,\overline{a})$, so that the fibre $f^{-1}(\overline{b}_i)$ is defined by some $\psi_i(\overline{x},\overline{a}) \in \mathcal{D}_{n_1}$.*
- $\mathrm{tp}^{\mathcal{M}^+}(\overline{b}_0/\overline{a}_0),...,\mathrm{tp}^{\mathcal{M}^+}(\overline{b}_{k-1}/\overline{a}_0)$ *includes every complete type of $\mathcal{M}^+$ over $\overline{a}_0$ that is consistent with $p(\overline{x},\overline{a}_0)$.*

Then

$$d^+(\theta_0(\mathcal{M},\overline{a})) = d^+(\theta_1(\mathcal{M},\overline{a})) + \max_{i<k} d^+(\psi_i(\mathcal{M},\overline{a})).$$

Proof. Let $e_i = d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}\left(\psi_i(\mathcal{M}^*,f\overline{b}_i)\right)$ for each $i < k$. From item 4 of Lemma 2.8 of [CITE:Udi-oleron] and Lemmas 3.9 and 3.8 above, we know that

$$\begin{aligned} d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(\theta_0(\mathcal{M}^*,f\overline{a})) &= \sup\left\{\alpha+\beta : \alpha \in [0,\infty),\, \beta = d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}\left(\left\{\overline{b} : d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(f^{-1}(\overline{b})) \geq \alpha\right\}\right)\right\} \\ &= \sup\left\{e_i+\beta : i<k,\, \beta = d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}\left(\left\{\overline{b} : d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(f^{-1}(\overline{b})) \geq e_i\right\}\right)\right\} \\ &= \sup\left\{e_i + d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(\theta_1(\mathcal{M}^*,f\overline{a})) : i<k\right\} \\ &= d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(\theta_1(\mathcal{M}^*,f\overline{a})) + \max_{i<k} d^{\hat{\mathscr{U}}}_{\mathrm{Hru}}(\psi_i(\mathcal{M}^*,f\overline{a})) \end{aligned}$$

which suffices. $\square$

In Lemmas 3.7 and 3.8 and Observation 3.9, we have proven the following proposition, which completes the proof of Theorem 3.1.

Proposition 3.11. *The data $(\mathbf{K}^+, \hat{\mathbf{K}}(\mathcal{M}), \hat{\mathscr{U}}, f)$ constructed from a selection σ and associated ultrafilter $\mathscr{U}$ are admissible, so*

$$d : \mathsf{Def}(\mathcal{M}) \to \mathbb{R}\cup\{-\infty\} : X \mapsto d(X) = \max\left\{d^+(gX) : g \in Aut(\mathcal{M}^+)\right\}$$

is a manufactured dimension function.

4 Towards construction well-ordered dimension functions

In hindsight, the ideas exposed in this paper can be viewed as an approach to a certain well-known and longstanding open question on the interplay between pseudo-finiteness and generic constructions.

Remark 4.1. Let $\mathbf{H}$ be the class of finite triangle-free graphs (in the signature $\mathsf{sig}(\mathscr{L})$ consisting of one binary relation symbol), and let $\mathcal{H}$ be the generic model of $\mathbf{H}$, known as the Henson graph, the generic triangle-free graph, and (unfortunately) the random triangle-free graph. Then, the long-standing open question is, "*Is the Henson graph psuedo-finite?*" (This is an interesting test problem related to a somewhat more general question asked by G. Cherlin in [1]: "When does a homogeneous structure for a finite relational language have the finite model property? More broadly, is there anything of interest in graph theory besides randomness and algebra?")

Although $\mathcal{H}$ is mysterious in many ways, we can make/collect a number of observations that may have bearing on the pseudo-finiteness of $T_{\mathbf{H}}$:

- If $T_{\mathbf{H}} = Th(\mathcal{H})$ were pseudo-finite, then since $(T_{\mathbf{H}})_\forall$ is finitely-axiomatizable and $\mathcal{H}$ is $\mathbf{H}$-universal and ultrahomogeneous, $T_{\mathbf{H}}$ would necessarily have the finite sub-model property.
- $\mathbf{H}$ is Ramsey-expandable. In fact, writing $\mathbf{H}^<$ for the class of ordered finite triangle-free graphs (i.e. A such that $<^A$ is a linear ordering of A and $A{\restriction}\mathscr{L} \in \mathbf{H}$), it was proved in [21], then $\mathbf{H}^<$ is a Ramsey class.
- $T_{\mathbf{H}}$ is not a simple theory, but since it is algebraically trivial and weakly eliminates imaginaries, it is super-rosy of $U^{\text{þ}}$-rank 1.

Thus:

- *If $T_{\mathbf{H}}$ were pseudo-finite, then by the Ramsey-expandability of $\mathbf{H}$ and Theorem 3.1, $\mathcal{H}$ would admit manufactured dimension functions.*
- *Since $T_{\mathbf{H}}$ is not simple, none of these manufactured dimension functions could have well-ordered images.*
- *If one could prove that for every Ramsey-expandable tame class $\mathbf{K}$, if $T_{\mathbf{K}}$ is super-rosy of finite rank, then it has well-ordered manufactured dimension functions, then it would follow that $T_{\mathbf{H}}$ is not pseudo-finite.*

Once again, we fix a strongly coherent tame class $\mathbf{K}$ with $\hat{\mathbf{K}}$ as a co-final super-robust subclass. We also fix a respectful expansion $\mathbf{K}^+$ of $\mathbf{K}$ that has the Ramsey property. $\mathcal{M}^+$ is the generic model of $\mathbf{K}^+$, and $\mathcal{M} = \mathcal{M}^+{\restriction}\mathscr{L}$ is the generic model of $\mathbf{K}$.

- Let $\mathbb{S} = \mathbb{S}_{\mathbf{K}:\mathbf{K}^+} =$

  $$\left\{\sigma \in \prod_{f \in F_{\text{dim}}} \{0, ..., k_f - 1\} : \mathscr{W}_\sigma \text{ has the finite intersect. property}\right\}$$

  where

  $$\mathscr{W}_\sigma = \{X \subseteq \mathbf{K}(\mathcal{M}) : (\exists n, G_0, A)\, Z(F_0, G_0; \sigma{\restriction}F_0) \cap \mathbf{K}_A(\mathcal{M}) \subseteq X\}$$

  for each $\sigma \in \prod_{f \in F_{\text{dim}}} \{0, ..., k_f - 1\}$.

  It can be shown that $\mathbb{S}$ is a closed G_δ subset of the full Cantor space $\prod_{f \in F_{\text{dim}}} \{0, ..., k_f - 1\}$.

- For each r, let $F_{\dim}(1,r) = \{f_{\varphi,r} : \varphi \in \mathcal{D}_1\}$, and for $F_0 \subset F_{\dim}(1,r)$ and $Q \subset_{\mathrm{fin}} \mathbb{Q} \cap [0,1]$, let

  $$V(F_0;Q) = \left\{ \sigma \in \mathbb{S} : \bigwedge_{f \in F_0} \left[\sigma_f \cdot 2^{-r}, (\sigma_f + 1)\cdot 2^{-r}\right] \cap Q \neq \emptyset \right\}$$

  which is a clopen subset of $\mathbb{S}$.

- Then $V_N =$

  $$\bigcap_{r \in \mathbb{N}^+} \bigcap_{F_0 \subset_{\mathrm{fin}} F_{\dim}(1,r)} \bigcup_{Q \in \binom{\mathbb{Q}\cap[0,1]}{N}} V(F_0;Q)$$

  is G_δ set for each $N \in \mathbb{N}^+$, and $\mathbb{S}^{\mathrm{fin}} = \bigcup_{N \in \mathbb{N}^+} V_N$ is $\mathbf{\Sigma}^0_3$.

By Theorems 2.3, we know that if $\mathbb{S}^{\mathrm{fin}}$ is non-empty, then $T_{\mathbf{K}}$ is super-simple of finite D-rank. To the present author, it seems unlikely that $\mathbb{S}^{\mathrm{fin}}$ is non-empty but "very small." That is, we sense that super-simplicity in this context should itself be tantamount to something similar to N-dimension asymptotic-ness (in the sense [4]), and the following conjecture just formalizes this intuition.

Conjecture 4.2. *Let* $\mathbf{K}$ *be a strongly coherent tame class with a respectful expansion* $\mathbf{K}^+$ *that has the Ramsey property. Let* $\mathcal{M}^+$ *be the generic model of* $\mathbf{K}^+$ *and* $\mathcal{M} = \mathcal{M}^+ \restriction \mathscr{L}$*. Then,* $T_{\mathbf{K}}$ *is super-simple of finite* D*-rank if and only if* $\mathbb{S}^{fin}$ *is a dense subset of* $\mathbb{S}$.

If super-rosiness where already sufficient for the density of $\mathbb{S}^{\mathrm{fin}}$ in this context, it would settle the question of the pseudo-finiteness of the Henson graph. More generally, it would suggest that the "universal" dimension theory engendered by super-rosiness – apparently an order-theoretic notion (see [2]) – is compatible with the algebraic pseudo-finite dimension functions discussed in this paper, perhaps by way of [16]. (We ask for density of $\mathbb{S}^{\mathrm{fin}}$ because it does not appear that the rosiness' dimension theory is as universal as we might like.) The conjecture is the following:

Conjecture 4.3. *Let* $\mathbf{K}$ *be a strongly coherent tame class with a respectful expansion* $\mathbf{K}^+$ *that has the Ramsey property. Let* $\mathcal{M}^+$ *be the generic model of* $\mathbf{K}^+$ *and* $\mathcal{M} = \mathcal{M}^+ \restriction \mathscr{L}$*, and let* $\mathscr{U} \in S_{\mathsf{Cone}}(\mathcal{M})$*. Then, if* $T_{\mathbf{K}}$ *is super-rosy of finite* $U^{\flat}$*-rank, then* $\mathbb{S}^{fin}$ *is a dense subset of* $\mathbb{S}$.

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