AN OBSERVATION REGARDING 0,1-LAWS AND ASYMPTOTICS OF DEFINABLE SETS IN GEOMETRIC FRAÏSSÉ CLASSES

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ABSTRACT. We examine one consequence for the generic theory $T_{\mathbf{C}}$ of a geometric Fraïssé $\mathbf{C}$ when $\mathbf{C}$ has the 0,1-law for first-order logic with convergence to $T_{\mathbf{C}}$ itself. We show that in this scenario, if the asymptotic probability measure in play is not terribly exotic, then $\mathbf{C}$ is "very close" to being a 1-dimensional asymptotic class – so that $T_{\mathbf{C}}$ is super-simple of finite SU-rank.

Keywords: Fraïssé classes, Asymptotic classes, 0,1-Laws, Simple theories, Geometric theories.

Mathematics Subject Classifications: `03C13, 03C15, 03C45, 05A16`.

INTRODUCTION

In much of the existing work on 0,1-laws for Fraïssé classes $\mathbf{C}$, researchers have focused almost entirely on the most ordinary of asymptotic probability measures $\mu = (\mu_N)_N$ – namely, μ_N is the uniform probability measure on members of $\mathbf{C}$ with universe $N = \{0, 1, ..., N-1\}$. When there is a 0,1-law relative to such μ with $Th(\mu) = T_{\mathbf{C}}$[1], one very often finds that $T_{\mathbf{C}}$ was already a very special sort of theory. For example, if $\mathbf{G}$ is the class of all finite graphs, then $T_{\mathbf{G}}$, the theory of the random graph, is super-simple of SU-rank 1, and one finds something similar when $\mathbf{C}$ is the class of finite bipartite graphs, the class of finite partial orders of rank 2, finite directed trees of rank 2, and so on. There is a sense that if a class $\mathbf{C}$ has the 0,1-law in a way similar to the way that $\mathbf{G}$ does so, then geometrically speaking, $T_{\mathbf{C}}$ is very much like $T_{\mathbf{G}}$.

This discussion requires an answer to the question, "What does it mean for $\mathbf{C}$ to have the 0,1-law in a way similar to the way that $\mathbf{G}$ does so?" In this paper, we answer this question for geometric Fraïssé classes (i.e. $T_{\mathbf{C}}$ is a geometric theory) by focusing on (i) the conditional independence properties of the asymptotic probability measure μ, and (ii) the requirement that $Th(\mu) = T_{\mathbf{C}}$. We find that if μ is not too weird, then not only is $T_{\mathbf{C}}$ super-simple of finite rank (as is $T_{\mathbf{G}}$), but up to excluding a neglible part of $\mathbf{C}$, the sizes of definable sets in members of $\mathbf{C}$ are under very tight, uniform control in a sense proposed in [8, 3, 4], generalizing results of [2] for finite fields. Speaking formally, we prove the following theorem:

Theorem 2.1. *Let $\mathbf{C}$ be a geometric Fraïssé class, and suppose that $\mathbf{C}$ has the 0,1-law for first-order logic with $Th(\mu) = T_{\mathbf{C}}$ relative to an asymptotic probability measure μ that has almost-independent sampling. Then $\mathbf{C}$ contains a cofinal super-robust sub-class $\mathbf{D}$, which is a 1-dimensional asymptotic class.*

[1] $T_{\mathbf{C}}$ being the complete theory of the generic model, or Fraïssé limit, of $\mathbf{C}$.

Using theorems of [8, 4] and [6], we derive the following theorem as a corollary.

Theorem 2.2. *Let* $\mathbf{C}$ *be a geometric Fraïssé class, and suppose that* $\mathbf{C}$ *has the 0,1-law for first-order logic with* $Th(\mu) = T_{\mathbf{C}}$ *relative to an asymptotic probability measure* μ *that has almost-independent sampling. Then* $T_{\mathbf{C}}$ *is super-simple, and for every definable set* X *of the generic model* $\mathcal{M}$ *of* $\mathbf{C}$*,* $D(X)$ *is bounded by the algebraic dimension of* X*.*

Regarding certain open questions of the form "Does a certain geometric Fraïssé class $\mathbf{C}$ have the 0,1-law relative to some unknown measure?" there is another moral in this story: An asymptotic probability measure that yields a 0,1-law with convergence to an *un*simple $\aleph_0$-categorical theory would have to be profoundly exotic.

Outline of the paper. This paper consists of just two main sections (plus a few concluding thoughts). In the first section, we collect all of the relevant definitions, recalling the definitioins of Fraïssé classes, geometric theories, super-robust classes, and 1-dimensional asymptotic classes. We also review the definitions of asymptotic probability measures and 0,1-laws relative to them. Finally, we identify what is meant by a non-exotic measure: an asymptotic probability measure with almost-independent sampling. In the second section, we prove our two main theorems. In a third section, we note a few open questions related to the results presented here.

1. Definitions: Asymptotic probability measures, almost-independent sampling, and super-robustness

As already noted, in this section, we will introduce all of the definitions that are relevant throughout the paper. This includes Fraïssé classes, super-robustness, asymptotic classes, asymptotic probability measures, 0,1-laws, and almost-independent sampling. Before these definitions, however, we establish a few notational and terminological conventions as follows.

1.0.1. *Notational and terminological conventions.* Throughout this paper, we make certain demands on our languages, and we use a somewhat eccentric notation for finite structures. These are accounted for in the following two conventions.[2]

Convention. Throughout this article, any language $\mathscr{L}$ in question is the first-order language built over a countable signature $\mathrm{sig}(\mathscr{L})$ that has *no function symbols and only finitely many constant symbols.*

Given $\mathscr{L}$, $\mathscr{L}^p$ is the set of partitioned $\mathscr{L}$-formulas, usually written $\varphi(\overline{x}; \overline{y})$ – an ordinary $\mathscr{L}$-formula with a partition of the free variables understood (allowing that the tuple of parameter variables $\overline{y}$ may be empty). We write $\mathscr{L}^{\mathsf{qf}}$ for the set of quantifier-free $\mathscr{L}$-formulas. Also, if φ is a formula, then $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$.

Convention. Fix a language $\mathscr{L}$.

- $\mathscr{L}$-structures that *might be* infinite are rendered as uppercase calligraphic letters – as in $\mathcal{M}, \mathcal{N}, ...$ – and their universes are the corresponding uppercase Roman charactera, $M, N, ...$, respectively, with cardinalities $|M|, |N|, ...$ respectively.

[2]The presentation here is quite similar to that in [6], and indeed, there is some overlap. There are only so many ways one can review the basic material on Fraïssé classes.

- If $\mathcal{M}$ is an $\mathscr{L}$-structure and $A \subseteq M$, then $\mathcal{M}[A]$ denotes the induced substructure of $\mathcal{M}$ with universe, $A \cup \left\{c^{\mathcal{M}} : c, \text{ a constant of } \mathrm{sig}(\mathscr{L})\right\}$. We write $\mathcal{M} \leq \mathcal{N}$ to mean that $\mathcal{M}$ is an induced substructure of $\mathcal{N}$: $M \subseteq N$ and $\mathcal{N}[M] = \mathcal{M}$. Extending this notation somewhat, we write $\mathcal{M} \leq^* \mathcal{N}$ to mean that there is an embedding $\mathcal{M} \to \mathcal{N}$.
- We use lowercase gothic letters, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ to denote $\mathscr{L}$-structures that *are certainly finite*. The universe of $\mathfrak{a}$ is $\|\mathfrak{a}\|$, and $|\mathfrak{a}|$ is the cardinality of $\mathfrak{a}$ – i.e. $|\mathfrak{a}| = \big|\|\mathfrak{a}\|\big|$. The notations already mentioned pertain as well to finite structures.

In order to reduce the amount of qualifying verbiage in our discussions of classes of finite structures, we establish the following basic qualifications once and for all.

Convention. To say, "$\mathbf{C}$ is a class of finite structures," we require:

- All members of $\mathbf{C}$ are finite structures for the same language $\mathscr{L}$.
- $\mathbf{C}$ is closed under isomorphism: If $\mathfrak{a} \in \mathbf{C}$ and $\mathfrak{b} \cong \mathfrak{a}$, then $\mathfrak{b} \in \mathbf{C}$.
- $\mathbf{C}$ is infinite *modulo* isomorphisms.
- For every $0 < n < \omega$, the set $S_n^{\mathsf{qf}}(\mathbf{C}) = \left\{\mathrm{qftp}^{\mathfrak{a}}(\bar{b}) : \mathfrak{a} \in \mathbf{C}, \bar{b} \in \|\mathfrak{a}\|^n\right\}$ is finite. Here, $\mathrm{qftp}^{\mathfrak{a}}(\bar{b})$ denotes the quantifier-free-complete type of $\bar{b}$ in the sense of $\mathfrak{a}$.

Given some structure $\mathcal{N}$ (possibly finite), we write $\mathbf{C}(\mathcal{N})$ for the set $\{\mathfrak{a} \in \mathbf{C} : \mathfrak{a} \leq \mathcal{N}\}$.

Associated with any class of finite structures $\mathbf{C}$, there is a foundation rank $\mathrm{rk}^{\mathbf{C}}$, which may or may not be useful. (For super-robust classes, the foundation rank is fundamental; for asymptotic classes in general, the foundation rank doesn't play any role to speak of.)

Definition 1.1. Let $\mathbf{C}$ be a class of finite structures. Then we define its rank function $\mathrm{rk}^{\mathbf{C}} : \mathbf{C} \to \omega$ as follows:

- $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq 0$ for all $\mathfrak{a} \in \mathbf{C}$.
- $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq n+1$ if there is an $\mathfrak{a}_0 \in \mathbf{C}(\mathfrak{a}) \setminus \{\mathfrak{a}\}$ and $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq n$.

Convention. For $a, b \in [0, \infty)$ and $0 < \varepsilon < 1$, we write $a \in (1 \pm \varepsilon)b$ as shorthand for

$$(1-\varepsilon)b < a < (1+\varepsilon)b.$$

Also, if X is some set and $0 < n < \omega$, then $X^{(n)}$ is the set of $\bar{x} \in X^n$ such that $x_i \neq x_j$ for all $i < j < n$.

1.1. **Fraïssé and super-robust classes, cofinality of classes of structures.** Our next batch of definitions concerns classes of finite structures with additional properties that allow them to be canonically pieced together into a single countably infinite structure. The first examples of such classes noted in the literature (e.g. [5]) were Fraïssé classes, although related formulations are now widely studied under various names.

1.1.1. *Fraïssé classes and cofinality.*

Definition 1.2. Let $\mathbf{C}$ be a class of finite structures. We say that $\mathbf{C}$ is a Fraïssé class if it has the following three properties:

Joint-embedding (JEP): For any two $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathbf{C}$, there are $\mathfrak{b} \in \mathbf{C}$ and embeddings $\mathfrak{a}_0 \to \mathfrak{b}$ and $\mathfrak{a}_1 \to \mathfrak{b}$.

Amalgamation (AP): For $\mathfrak{a}, \mathfrak{b}_0, \mathfrak{b}_1 \in \mathbf{C}$ and embeddings $f_i : \mathfrak{a} \to \mathfrak{b}_i$ $(i < 2)$, there are $\mathfrak{c} \in \mathbf{C}$ and embeddings $f_i' : \mathfrak{b}_i \to \mathfrak{c}$ $(i < 2)$ such that $f_0' \circ f_0 = f_1' \circ f_1$.

Heredity (HP): $\mathfrak{a}[B] \in \mathbf{C}$ whenever $\mathfrak{a} \in \mathbf{C}$ and $B \subseteq \|\mathfrak{a}\|$.

Following the terminology of [6], a class that has JEP and AP is a *semi-coherent class*

Fraïssé classes, and semi-coherent classes more generally, are attractive precisely because AP and JEP engender the following proposition.

Proposition 1.3. *Let* $\mathbf{C}$ *be a semi-coherent class in a language* $\mathscr{L}$*. There is a countably infinite* $\mathscr{L}$*-structure* $\mathcal{M}$ *satisfying the following:*

1. *(*$\mathbf{C}$*-Universality) For every* $\mathfrak{a} \in \mathbf{C}$*, there is an embedding* $\mathfrak{a} \to \mathcal{M}$*.*
2. *(*$\mathbf{C}$*-Homogeneity) For every* $\mathfrak{a} \in \mathbf{C}$ *such that* $\mathfrak{a} \leq \mathcal{M}$*, for every embedding* $f : \mathfrak{a} \to \mathcal{M}$*, there is an automorphism* $g \in Aut(\mathcal{M})$ *such that* $f \subseteq g$*.*
3. *For every* $A \subset_{fin} M$*, there is a finite substructure* $\mathfrak{b} \leq \mathcal{M}$ *such that* $\mathfrak{b} \in \mathbf{C}$ *and* $A \subseteq \|\mathfrak{b}\|$*.*
4. $\mathcal{M}$ *is the prime model of its own theory.*

Furthermore, up to isomorphism, $\mathcal{M}$ *is the unique countable structure that satisfies items 1-3. Thus, we may speak of* the *generic model of a semi-coherent class* $\mathbf{C}$*, and the assignment* $T_{\mathbf{C}} := Th(\mathcal{M})$ *is well-defined.*

Obviously every Fraïssé class is a semi-coherent class, so one naturally asks if there is a characterization of Fraïssé-ness in terms of generic models and generic theories. The following proposition provides that characterization.

Proposition 1.4. *Let* $\mathbf{C}$ *be a semi-coherent class in a language* $\mathscr{L}$*, and let* $\mathcal{M}$ *be its generic model. The following are equivalent:*

1. $T_{\mathbf{C}}$ *is* $\aleph_0$*-categorical and eliminates quantifiers.*
2. $\mathrm{Age}(\mathcal{M}) = \{\mathfrak{a} : \mathfrak{a} \leq^* \mathcal{M}\} \supseteq \mathbf{C}$ *is a Fraïssé class.*

A Fraïssé class may contain finite structures that are pathological relative to the other structures in the class. For example, the class $\mathbf{G}$ of all finite graphs contains all of the cycles, but a large "typical" graph – typical member of $\mathbf{G}$ – is not remotely like a cycle. The following definition of cofinality for semi-coherent classes just gives us a framework for eliminating pathological members of a class without changing the class in any essential way.

Definition 1.5. Let $\mathbf{C}, \mathbf{D}$ be semi-coherent classes. We say that $\mathbf{C}$ and $\mathbf{D}$ are *cofinal* if the following are both true:

- For every $\mathfrak{c} \in \mathbf{C}$, there is a $\mathfrak{d} \in \mathbf{D}$ such that $\mathfrak{c} \leq \mathfrak{d}$.
- For every $\mathfrak{d} \in \mathbf{D}$, there is a $\mathfrak{c} \in \mathbf{C}$ such that $\mathfrak{d} \leq \mathfrak{c}$.

(Obviously, a cofinal subclass of $\mathbf{C}$ is a subclass $\mathbf{C}_0 \subseteq \mathbf{C}$ such that $\mathbf{C}$ and $\mathbf{C}_0$ are cofinal.) We note that if $\mathbf{C}$ and $\mathbf{D}$ are cofinal semi-coherent classes, then $T_{\mathbf{C}} = T_{\mathbf{D}}$, and the converse holds as well.

1.1.2. *Super-robustness.* In general, first-order properties and first-order definable sets of the generic model and those of members of a semi-coherent class $\mathbf{C}$ need not coincide. (Think of $\mathbb{Q}$ and the class of finite linear orders, say.) In [9], and then in [6], the notion of *robustness* of a chain or class of finite structures was introduced; in both cases, one introduces a graded, approximate form of the elementary substructure relation.

Definition 1.6. Let $\mathbf{C}$ be a semi-coherent class of finite $\mathscr{L}$-structures. We say that $\mathbf{C}$ is super-robust if there is a function $\nu : \mathscr{L} \to \omega$ such that for all $\mathfrak{a}, \mathfrak{b} \in \mathbf{C}$, $\varphi(x_0, ..., x_{n-1}) \in \mathscr{L}$, and $\overline{a} \in \|\mathfrak{a}\|^n$, if $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq \nu(\varphi)$ and $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{a} \models \varphi(\overline{a}) \Leftrightarrow \mathfrak{b} \models \varphi(\overline{a})$.

In [6], a large number of equivalents of super-robustness are examined. In this paper, we will have need of one of these. (Eventually, we will use it to transfer the super-simplicity of ultraproducts over a derived 1-dimensional asymptotic class back to the generic theory of a given Fraïssé class.)

Theorem 1.7. *Let* $\mathbf{C}$ *be a Fraïssé class with generic model* $\mathcal{M}$*, and let* $\mathbf{D}$ *be a super-robust cofinal subclass of* $\mathbf{C}$*. Let* $\mathfrak{d}_0, ..., \mathfrak{d}_n, ...$ *be any enumeration of (representatives of) isomorphism types in* $\mathbf{D}$*, and let* $\mathscr{U}$ *be any non-principal ultrafilter on* ω*. Then* $\prod_n \mathfrak{d}_n/\mathscr{U} \models T_{\mathbf{C}}$*.*

1.2. **Geometric theories.** The formulation of asymptotic classes (below) asks for both dimension and measure of definable sets in finite structures. For a general Fraïssé class, it's not at all clear where to find a "canonical" dimension theory. (We note that pseudo-finite dimensions, as in [7], can vary wildly depending on the ultrafilter chosen.) In this paper, we dispense with this issue by focusing attention on Fraïssé classes in which *there is* a natural dimension theory – those classes for which the generic theory is geometric in the following sense.

Definition 1.8. Let T be a complete theory with infinite models. We say that T is a geometric theory if both of the following hold:

- T eliminates $\exists^\infty$: For every $\varphi(\overline{x};\overline{y})$, there is a number n_φ such that for all $\mathcal{M} \models T$ and $\overline{b} \in M^{|\overline{y}|}$, if $\mathcal{M} \models \exists_{\geq n_\varphi}\overline{x}\,\varphi(\overline{x},\overline{b})$, then $\varphi(\mathcal{M},\overline{b})$ is infinite. (This condition is, of course, immediate for all $\aleph_0$-categorical theories in countable languages.)
- For every $\mathcal{M} \models T$, $\mathrm{acl}^{\mathcal{M}}$ is the closure operator of a pre-geometry on M – meaning that for all $C \subseteq M$ and $a, b \in M$, if $a \in \mathrm{acl}^{\mathcal{M}}(bC) \setminus \mathrm{acl}^{\mathcal{M}}(C)$, then $b \in \mathrm{acl}^{\mathcal{M}}(aC)$.

For our purpses, then, a Fraïssé class $\mathbf{C}$ is geometric just in case $T_{\mathbf{C}}$ is a geometric theory, and if $\mathbf{C}$ is a geometric Fraïssé class, then we write $\mathbf{C}^{\mathfrak{a}}$ for the subclass consisting of those $\mathfrak{a} \in \mathbf{C}$ for which there is an embedding $f : \mathfrak{a} \to \mathcal{M}$, where $\mathcal{M} \models T_{\mathbf{C}}$, such that $\mathrm{acl}^{\mathcal{M}}(\|f\mathfrak{a}\|) = \|f\mathfrak{a}\|$. (By quantifier-elimination in $T_{\mathbf{C}}$, algebraic-closedness is an invariant of the isomorphism type of a member of $\mathbf{C}$.)

Once a theory is known to be geometric, we also know, as with strongly-minimal theories, that cardinalities of algebraically-independent subsets are invariants, and this provides us with a rudimentary dimension theory that extends easily to a dimension theory for definable sets as follows.

Definition 1.9. If T is a geometric theory, then every model $\mathcal{M} \models T$ admits a dimension function of the form,

$$\dim_{\mathbf{a}}(\overline{a}/B) = \max\left\{ S \subseteq n : \begin{array}{l} |\{a_i\}_{i\in S}| = |S|, \\ \{a_i\}_{i\in S} \text{ is alg. ind. over } B \end{array} \right\}$$

for $0 < n < \omega$, $\overline{a} \in M^n$, and $B \subseteq M$. One extends this rudimentary dimension function to a dimension function on definable sets: for a definable set $X \in \mathsf{Def}(\mathcal{M})$,

$$\dim_{\mathbf{a}}(X) = \min\left\{ \max\left\{\dim_{\mathbf{a}}(\overline{a}/B) : \overline{a} \in X\right\} : \begin{array}{c} B \subset_{\mathrm{fin}} M, \\ X \text{ is over } B \end{array} \right\}.$$

Everything in the sequel is applied more or less exclusively to geometric Fraïssé classes, so it will be convenient (for succinctness) to establish the following blanket assumption.

Fixed from now on... *Unless stated otherwise,* $\mathbf{C}$ *denotes a* geometric *Fraïssé class, and* $\mathcal{M}$ *is its generic model.*

1.3. **Asymptotic probability measures, 0,1-laws, and almost-independent sampling.** In this subsection, we finally formalize the notions of asymptotic probability measures and 0,1-laws relative to them for geometric Fraïssé classes. The first definition below accounts for asymptotic probability measures for geometric Fraïssé classes, and in the subsequent definition, we identify a few notational tricks that are needed for our formulation of 0,1-laws for first-order logic which is given in Definition 1.13. We note that our definitions do not require that $\mathscr{L}$'s signature is finite, only that $\mathbf{C}_N$ is finite for every N.

Definition 1.10. For every $0 < N < \omega$, $\mathbf{C}_N$ is the set $\{\mathfrak{a} \in \mathbf{C} : \|\mathfrak{a}\| = N\}$ and $\mathbf{C}_N^a = \mathbf{C}_N \cap \mathbf{C}^a$. Now, an *asymptotic probability measure for* $\mathbf{C}$ is a sequence $\mu = (\mu_N)_N$ such that:

- For every $0 < N < \omega$, $\mu_N : \mathbf{C}_N \to [0,1]$ is an isomorphism-invariant probability mass function.
- For all $N < N_1 < \omega$, for all $\mathfrak{a} \in \mathbf{C}_N^a$, $\mu_N(\mathfrak{a}) = \sum_{\mathfrak{b} \in \mathbf{C}_{N_1} : \mathfrak{a} \leq \mathfrak{b}} \mu_{N_1}(\mathfrak{b})$.

For $X \subseteq \mathbf{C}_N$, we write $\mu_N X$ or $\mu_N[X]$ as shorthand for $\sum_{\mathfrak{a} \in X} \mu_N(\mathfrak{a})$.

Definition 1.11. Let μ be an asymptotic probability measure for $\mathbf{C}$, and let $0 < N < \omega$.

- For a sentence $\varphi \in \mathit{Sent}(\mathscr{L}(N))$, we define
$$\mu_N[\varphi] = \sum_{\mathfrak{a} \in \mathbf{C}_N : \mathfrak{a} \vDash \varphi} \mu_N(\mathfrak{a}) = \mu_N \{\mathfrak{a} \in \mathbf{C} : \mathfrak{a} \vDash \varphi\}$$
where "$\mathfrak{a} \vDash \varphi$" is given its natural meaning.
- Suppose $\theta = \theta(\bar{b})$ and $\varphi = \varphi(\bar{a}, \bar{b})$ are sentences in $\mathscr{L}(N)$, and suppose $\mu_N[\theta] > 0$. Then
$$\mu_N[\varphi|\theta] = \frac{\mu_N[\varphi \wedge \theta]}{\mu_N[\theta]}$$

Furthermore, we define a partial function $\mu_\infty : \mathit{Sent}(\mathscr{L}(\omega)) \rightharpoonup [0,1]$ by
$$\mu_\infty[\varphi] = \begin{cases} \lim_{N\to\infty} \mu_N[\varphi] & \text{if the limit exists} \\ \uparrow & \text{otherwise} \end{cases}$$

Observation 1.12. Let μ be an asymptotic probability measure for $\mathbf{C}$.

- If θ is a *quantifier-free* sentence in $\mathscr{L}(\omega)$, then $\mu_\infty[\theta]$ exists.
- Suppose $\theta = \theta(\bar{b})$, $\varphi = \varphi(\bar{a}, \bar{b})$ are quantifier-free sentences in $\mathscr{L}(\omega)$, and suppose $\mu_\infty[\theta] > 0$. Then the limit $\mu_\infty[\varphi|\theta] = \lim_{N\to\infty} \mu_N[\varphi|\theta]$ exists, and $\mu_\infty[\varphi|\theta] = \mu_\infty[\varphi \wedge \theta]/\mu_\infty[\theta]$.

Definition 1.13. Let μ be an asymptotic probability measure for $\mathbf{C}$. We define,
$$Th(\mu) = \{\varphi \in \mathit{Sent}(\mathscr{L}) : \mu_\infty(\varphi)\downarrow = 1\}.$$

Ordinarily, $\mathbf{C}$ would be said to have the 0,1-law for first-order logic relative to μ just in case $Th(\mu)$ is a complete theory, but in this paper, it will be more convenient to fix a stronger definition. We say that $\mathbf{C}$ *is said to the 0,1-law for first-order logic relative to* μ if $Th(\mu) = T_{\mathbf{C}}$.

We must note that our last requirement in the specification of a 0,1-law – that $Th(\mu) = T_{\mathbf{C}}$ – is not a triviality; the class of finite triangle-free graphs provides a good example.

Remark 1.14. Let $\mathbf{C}$ be the class of finite triangle-free graphs in the language whose signature consists of a single binary relation symbol, and let $\mu = (\mu_N)_N$ be such that for every N, μ_N is the uniform distribution on $\mathbf{C}_N$. Then $Th(\mu)$ is a complete theory (the theory of the generic bipartite graph), but $Th(\mu) \neq T_{\mathbf{C}}$.

Now, we turn to the idea of *almost-independent sampling*, to which we will appeal in identifying measures of definable sets (relative to their algebraic dimensions). To motivate the definition, we note that in generating a random member of $\mathbf{G}_N$ by independent unbiased coin flips, the event of any particular vertex $i \in N \setminus C$ satisfying the requirements of an extension axiom (over some clump C of vertices) is independent of the event of any *other* vertex $j \in N \setminus C$ satisfying the same extension axiom. In [10] (exposed more tractably in [1]), this observation is mined to prove that typical finite graphs satisfy *strong* extension axioms, which are very similar to the conclusion established in our Proposition 2.3 below.

$\mathbf{G}$ is a geometric Fraïssé class, but the algebraic closure operation in $T_{\mathbf{G}}$ is trivial. To extend the idea of independent sampling to an arbitrary geometric Fraïssé class, we need to tailor the discussion a little more carefully. The basic objects, then, are d, n-extension problems, which are used both in the formulation of almost-independent sampling and heavily in the rest of the paper.

Definition 1.15. We define a *d, n-extension problem of* $\mathbf{C}$ to be a pair (φ, θ) where $\theta(\overline{y}) \in S_m^{\mathsf{qf}}(T_{\mathbf{C}})$, $\varphi(\overline{x}, \overline{y}) \in S_{d+n}^{\mathsf{qf}}(T_{\mathbf{C}})$, and together these satisfy:

- $\theta(\overline{y}) \vDash \bigwedge_{i<j<n} y_i \neq y_j$
- $\varphi(\overline{x}, \overline{y}) \vDash \theta(\overline{y}) \wedge \bigwedge_{i<j<d} x_i \neq x_j$.
- If $\mathcal{M} \vDash \varphi(\overline{a}, \overline{b})$, then $\overline{a} \cap \operatorname{acl}(\overline{b}) = \emptyset$ and $\{a_0, ..., a_{d-1}\}$ is algebraically independent over $\overline{b}$.

As it is slightly easier to typeset (and slightly more illustrative), we often write φ/θ in place of (φ, θ) for a d, n-extension problem of $\mathbf{C}$.

Definition 1.16. Let μ be an asymptotic probability measure for $\mathbf{C}$. We say that μ *has almost-independent sampling* if for every $1, n$-extension problem φ/θ of $\mathbf{C}$, there are numbers $0 \leq \varepsilon < 1$ and $0 < \delta < 1$ such that for almost every N, for every $\overline{b} \in N^{(n)}$, for any $\{i_0 < \cdots < i_{t-1}\} \subseteq N \setminus \overline{b}$ where $t = \lceil \delta N \rceil$, for every $s : t \to 2$,

$$\mu_N\Big[\bigwedge_{j<t} \varphi(i_j, \overline{b})^{s(j)} \,\Big|\, \theta(\overline{b})\Big] \in (1 \pm \varepsilon)\cdot \prod_{j<t} \mu_N\Big[\varphi(i_j, \overline{b})^{s(j)} \,\Big|\, \theta(\overline{b})\Big].$$

1.4. **Asymptotic classes.** The last definition required for this paper is that of an asymptotic class. These were introduced in [8, 3, 4], generalizing the situation for finite fields as proved in [2].

Definition 1.17. Let $\mathbf{D}$ be a class of finite structures in a countable language $\mathscr{L}$. We say that $\mathbf{D}$ is a *1-dimensional asymptotic class* if there is a function

$$\mathbb{A} : \mathscr{L}^p \to \mathcal{P}_{\text{fin}}\big(\mathscr{L} \times \omega \times [0, \infty)\big) : \varphi(\overline{x}; \overline{y}) \mapsto \mathbb{A}(\varphi)$$

such that for all $\varphi(\overline{x}; \overline{y}) \in \mathscr{L}^p$:

- $\mathbb{A}(\varphi) \neq \emptyset$, and for every $(\theta, d, m) \in \mathbb{A}(\varphi)$, θ is of the form $\theta(\overline{y})$.
- $\bigvee_{(\theta,d,m)\in\mathbb{A}(\varphi)} \theta(\overline{y}) \equiv \mathsf{True}$ *modulo* $\mathbf{D}$.

- For (θ_1, d_1, m_1) and (θ_2, d_2, m_2) in $\mathbb{A}(\varphi)$, if $\theta_1 \wedge \theta_2 \not\equiv \mathsf{False}$ *modulo* $\mathbf{D}$, then $d_1 = d_2$ and $m_1 = m_2$.
- For every $\varepsilon > 0$, there is an $N_{\varphi,\varepsilon} < \omega$ such that for all $\mathfrak{a} \in \mathbf{D}$, $(\theta, d, m) \in \mathbb{A}(\varphi)$, and $\overline{b} \in \|\mathfrak{a}\|^{|\overline{y}|}$, if $\mathfrak{a} \models \theta(\overline{b})$ and $|\mathfrak{a}| \geq N_{\varphi,\varepsilon}$, then

$$\left| \left|\varphi(\overline{a}, \overline{b})\right| - m|\mathfrak{a}|^d \right| < \varepsilon |\mathfrak{a}|^d.$$

(Clearly, it follows that if $(\theta, d, m) \in \mathbb{A}(\varphi(x_0, ..., x_{r-1}; \overline{y}))$, then $d \leq r$.)

For any number $0 < K < \omega$, we recover the notion of a *K-dimensional asymptotic class* by simply replacing ω in the range of $\mathbb{A}$ by $\{k/K : k < \omega\} \subseteq \mathbb{Q} \cap [0, \infty)$. Thus, for a K-dimensional asymptotic class, if $(\theta, k/K, m) \in \mathbb{A}(\varphi(x_0, ..., x_{r-1}; \overline{y}))$, then $k \leq Kr$.

We conclude this section by noting an important theorem showing that the infinitary model theory of ultraproducts over an asymptotic class is quite tame. We will also use this theorem to derive Theorem 2.2 from Theorem 2.1

Theorem 1.18 ([3])**.** *Let $\mathbf{D}$ be a class of finite structures in a countable language $\mathscr{L}$. If $\mathbf{D}$ is a K-dimensional asymptotic class, then the theory of any infinite ultra-product $\mathcal{M}$ of members of $\mathbf{D}$ is super-simple, and if $\mathcal{M}$ is geometric, then for any definable set $X \in \mathsf{Def}(\mathcal{M})$, $D(X) \leq K \cdot \mathrm{ari}(X)$.*

2. Demonstrations

In this section, of course, we prove the two main theorems of this paper. In the first, we show that a geometric Fraïssé class that has the 0,1-law for first-order logic relative to a "non-exotic" asymptotic probability measure is necessarily very close to being an asymptotic class. An important step in the proof of Theorem 2.1 is the "tail inequality" encoded as Proposition 2.3; we use the lower bounds resulting from Proposition 2.3 to identify measures of definable sets relative to their dimensions. Using Theorem 2.1, it is then a fairly simple matter to prove Theorem 2.2 using Theorems 1.18 and 1.7

Theorem 2.1. *Let $\mathbf{C}$ be a geometric Fraïssé class, and suppose that $\mathbf{C}$ has the 0,1-law for first-order logic relative to an asymptotic probability measure μ that has almost-independent sampling. Then $\mathbf{C}$ contains a cofinal super-robust sub-class $\mathbf{D}$, which is a 1-dimensional asymptotic class via $\mathbb{A}$, with*

$$\mathbb{A}(\varphi(\overline{x}; \overline{y})) \subset_{\mathit{fin}} \left\{ \left(\mathrm{qftp}(\overline{a}), \dim_{\mathbf{a}}\left(\varphi(\mathcal{M}; \overline{a})\right), m\right) : \overline{a} \in M^{|\overline{y}|},\ m \in [0, \infty) \right\}.$$

Theorem 2.2. *Let $\mathbf{C}$ be a geometric Fraïssé class, and suppose that $\mathbf{C}$ has the 0,1-law for first-order logic relative to an asymptotic probability measure μ that has almost-independent sampling. Then $T_{\mathbf{C}}$ is super-simple, and for every definable set X of the generic model $\mathcal{M}$ of $\mathbf{C}$, $D(X) \leq \dim_{\mathbf{a}}(X)$.*

2.1. A tail inequality.

The focus of this subsection is to prove Proposition 2.3 below. Our proof draws heavily on a related argument given in [1]. The starting point for our proof (and theirs) is the following standard result from discrete probability theory.

Fact (Chernoff bound)**.** *For any $0 < p, \alpha < 1$, there exists $0 < c < 1$ such that the following holds: For any $N < \omega$, if $X \sim \mathsf{Binomial}(N, p)$, then $\mathbb{P}\big[\, X \leq \alpha \cdot p \cdot N \,\big] \leq c^N$.*

An alternative statement, which is a little more convenient for us, is the following: There is a function $\xi : (0,1)\times(0,1) \to (0,1)$ such that the following holds. For any $0 < p, \alpha < 1$, for any $N < \omega$, if $X \sim \mathsf{Binomial}(N,p)$, then $\mathbb{P}\big[\, X \leq \alpha{\cdot}p{\cdot}N \,\big] \leq \xi(p,\alpha)^N$.

Proposition 2.3. *Let $\mathbf{C}$ be a geometric Fraïssé class, and suppose that $\mathbf{C}$ has the 0,1-law for first-order logic relative to an asymptotic probability measure μ that has almost-independent sampling. Then, for every d,n-extension problem φ/θ of $\mathbf{C}$, there is a number $0 < \beta < 1$ such that*

$$\mu_N \left\{ \mathfrak{a} : (\exists\, \overline{b} \in \theta(\mathfrak{a}))\, \left|\varphi(\mathfrak{a},\overline{b})\right| \leq \beta|\mathfrak{a}|^d \right\} \in O(\mathsf{poly}(N)c^N)$$

for some $0 < c < 1$.

Proof. It is not hard to see that the statement for arbitrary $0 < d < \omega$ follows by a straightforward induction from the case of $d = 1$. So here, we prove only the statement when $d = 1$. That is, we prove just that, *"For every $1,n$-extension problem φ/θ of $\mathbf{C}$, there is a number $0 < \beta \leq 1$ such that $\mu_N \left\{ \mathfrak{a} : (\exists\, \overline{b} \in \theta(\mathfrak{a}))\, \left|\varphi(\mathfrak{a},\overline{b})\right| \leq \beta|\mathfrak{a}| \right\} \in O(\mathsf{poly}(N)c^N)$."*

Claim. Let φ/θ be a $1,m$-extension problem of $\mathbf{C}$. There is a constant $0 < c < 1$ such that for sufficiently large N, for each $\overline{b} \in N^{(n)}$,

$$\mu_N \left[\left|\varphi(*,\overline{b})\right| \leq \frac{1}{2}p\,\delta N \;\middle|\; \theta(\overline{b}) \right] \leq c^N$$

where $p = (1-\varepsilon)\mu_N\big[\varphi(n;0,...,n-1)\big|\theta(0,...,n-1)\big]$, and $\mu_N \left[\left|\varphi(*,\overline{b})\right| \leq \frac{1}{2}p\,\delta N \;\middle|\; \theta(\overline{b}) \right]$ is shorthand for the conditional probability,

$$\frac{\mu_N \left\{ \mathfrak{a} : \mathfrak{a} \vDash \theta(\overline{b}) \wedge \left|\varphi(\mathfrak{a},\overline{b})\right| \leq \frac{1}{2}p\,\delta N \right\}}{\mu_N[\theta(\overline{b})]}$$

To prove the claim, it is helpful to isolate one straightforward observation on ensembles of $\{0,1\}$-valued random variables.

Observation. Suppose $0 < p < 1$ and $0 < \alpha < 1$, and for some $K_0 < \omega$, let $Y_0 \sim \mathsf{Binomial}(K_0,p)$. Let $K_0 \leq K < \omega$, and let $q : \{0,1\}^K \to [0,1]$ be a probability mass function that is invariant under permutations of coordinates and such that for all $\ell \leq K_0$ and $\{i_0 < \cdots < i_{\ell-1}\} \subseteq K$,

$$\mathbb{P}_q \left\{ \overline{x} \in \{0,1\}^K : \sum\nolimits_{j<K_0} x_{i_j} \geq \ell \right\} \geq \binom{K_0}{\ell} p^\ell (1-p)^{K_0-\ell}$$

Then

$$\mathbb{P}_q \left\{ \overline{x} \in \{0,1\}^K : \sum\nolimits_{j<K} x_{i_j} \leq \alpha{\cdot}p{\cdot}K_0 \right\} \leq \mathbb{P}\big[\, Y_0 \leq \alpha{\cdot}p{\cdot}K_0 \,\big].$$

Proof of claim. Let $K = N - n$ and $K_0 = \lceil \delta N \rceil$ (so for large enough N, $K_0 < K$), and define $q : \{0,1\}^K \to [0,1]$ by

$$q(\overline{x}) = \mu_N \left[\bigwedge_{i<K} \varphi(i;0,...,n-1)^{x_i} \;\middle|\; \theta(0,...,n-1) \right].$$

Also, let $X \sim \mathsf{Binomial}(K_0, p)$. Then, for each $\bar{b} \in N^{(n)}$ individually,

$$\begin{aligned}\mu_N\left[\left|\varphi(*,\bar{b})\right| \leq \frac{1}{2}p\,\delta N \,\middle|\, \theta(\bar{b})\right] &= \mathbb{P}_q\left\{\bar{x} \in \{0,1\}^K : \sum\nolimits_{j<K} x_{i_j} \leq \frac{1}{2}\cdot p\cdot \delta\cdot K_0\right\} \\ &\leq \mathbb{P}\left[\, X \leq \frac{1}{2}\cdot p\cdot\delta\cdot K_0\,\right] \\ &\leq \xi(p,\delta/2)^{\delta N}\end{aligned}$$

so we set $c = \xi(p,\delta/2)^{\delta}$. □

The proof of the proposition now concludes with just a little more calculation:

$$\begin{aligned}\mu_N\left[\bigvee_{\bar{b}\in N^{(n)}} \theta(\bar{b}) \wedge \left|\varphi(*,\bar{b})\right| \leq \frac{1}{2}p\,\delta N\right] &\leq \sum_{\bar{b}\in N^{(n)}} \mu_N\left[\theta(\bar{b}) \wedge \left|\varphi(*,\bar{b})\right| \leq \frac{1}{2}p\,\delta N\right] \\ &= \sum_{\bar{b}\in N^{(n)}} \mu_N\left[\left|\varphi(*,\bar{b})\right| \leq \frac{1}{2}p\,\delta N \,\middle|\, \theta(\bar{b})\right]\mu_N\left[\theta(\bar{b})\right] \\ &= \mu_N\left[\theta(0,...,n-1)\right] \sum_{\bar{b}\in N^{(n)}} \mu_N\left[\left|\varphi(*,\bar{b})\right| \leq \frac{1}{2}p\,\delta N \,\middle|\, \theta(\bar{b})\right] \\ &\leq \left(\mu_N\left[\theta(0,...,n-1)\right]\cdot n!\cdot\binom{N}{n}\right)\, c^N.\end{aligned}$$

□

2.2. **Proof of Theorems 2.1 and 2.2.** From this point, the remaining work in proving Theorem 2.1 consists in the following:

- Identify what the dimension-relative measures of definable sets *should* be in the 1-dimensional asymptotic subclass that we eventually extract.
- Extract that subclass.

In Definition 2.4 and Lemma 2.5, we use Proposition 2.3 to recover measures for d,n-extension problems, and in Definition 2.6, we state how one can form measures for arbitrary definable sets using those of extension problems (we take it as clear that this method works as advertised), completing the specification $\mathbb{A}$ for the 1-dimensional asymptotic class.

Definition 2.4. Let φ/θ be a d,n-extension problem of $\mathbf{C}$ for some $0 < d < \omega$ and $n < \omega$. Then we define:

$$\begin{aligned}U(\varphi/\theta) &= \left\{\beta \in [0,1] : (\exists\,\alpha{\in}(0,1))(\text{a.e.}N)(\forall\,\bar{b}{\in}N^{(n)})\ \mu_N\left[\left|\varphi(*,\bar{b})\right| \leq \beta N^d \,\middle|\, \theta(\bar{b})\right] \leq \alpha^N\right\}, \\ \beta(\varphi/\theta) &= \sup U(\varphi/\theta)\end{aligned}$$

Now, for $n < \omega$, $0 < d < \omega$, and $\theta(\bar{y}) \in S_n^{\mathsf{qf}}(T_{\mathbf{C}})$, let $J_d(\theta)$ be the set of all d,n-extension problems of $\mathbf{C}$ of the form φ/θ. Then for each $\varphi/\theta \in J_d(\theta)$, we define

$$\gamma(\varphi/\theta) = \frac{1}{2}\left(\beta(\varphi/\theta) + \left(1 - \sum_{\psi/\theta\in J_d(\theta):\psi\not\equiv\varphi} \beta(\psi/\theta)\right)\right).$$

Lemma 2.5. *Let φ/θ be a d,n-extension problem of $\mathbf{C}$ for some $0 < d < \omega$ and $n < \omega$. Then for every $0 < \varepsilon < 1$, there is a number $0 \leq c(\theta,\varepsilon) < 1$ such that for every large enough N and every $\overline{b} \in N^{(n)}$ and,*

$$\mu_N\left[\left|\left|\varphi(*,\overline{b})\right| - \gamma(\varphi/\theta)N^d\right| > \varepsilon N^d \,\middle|\, \theta(\overline{b})\right] \leq c(\theta,\varepsilon)^N.$$

Proof. In the first place, we note that if $J_d(\theta) = \{\varphi/\theta\}$ is a singleton, then $\gamma(\varphi/\theta) = 1$, and the claim of the lemma is immediate. So, let us assume that $J_d(\theta) = \{\varphi_0/\theta{=}\varphi/\theta, \varphi_1/\theta, ..., \varphi_{r-1}/\theta\}$ for some $r > 1$.

Let $\varepsilon > 0$ be given. Towards a contradiction, suppose that for every $0 < c < 1$, there are infinitely many N such that

$$\mu_N\left[\left|\varphi_0(*,\overline{b})\right| - \gamma(\varphi_0/\theta)N^d > \varepsilon N^d \,\middle|\, \theta(\overline{b})\right] > c^N$$

where $\overline{b} \in \omega^{(n)}$ is fixed arbitrarily. By the Pigeonhole Principle (and fiddling with monotonicities), it follows that there is a single $0 < i < r$ such that for every $0 < c < 1$, there are infinitely many N such that

$$\mu_N\left[\left|\varphi_i(*,\overline{b})\right| < \left(1 - \tfrac{\varepsilon}{2r+1}\right)\beta(\varphi_i/\theta)N^d \,\middle|\, \theta(\overline{b})\right] > c^N$$

which contradicts the definition of $\beta(\varphi_i/\theta)$. Thus, there is a number $0 < c^+(\theta,\varepsilon) < 1$ such that

$$\mu_N\left[\left|\varphi_0(*,\overline{b})\right| - \gamma(\varphi_0/\theta)N^d \leq \varepsilon N^d \,\middle|\, \theta(\overline{b})\right] \leq c^+(\theta/\varepsilon)^N$$

for almost all N. Arguing in largely the same way, we also recover a number $0 < c^-(\theta,\varepsilon) < 1$ such that

$$\mu_N\left[\gamma(\varphi_0/\theta)N^d - \left|\varphi_0(*,\overline{b})\right| \leq \varepsilon N^d \,\middle|\, \theta(\overline{b})\right] \leq c^-(\theta/\varepsilon)^N$$

for all but finitely many N, and we set $c(\theta,\varepsilon) = \max\{c^+(\theta,\varepsilon), c^-(\theta,\varepsilon)\}$ to complete the proof. $\square$

Definition 2.6. Let $\psi(x_0, ..., x_{k-1}; y_0, ..., y_{n-1}) \in \mathscr{L}^p$.

- Let $\psi^{\mathsf{qf}}(\overline{x};\overline{y})$ be the disjunction of all quantifier-free-complete types $q(\overline{x},\overline{y}) \in S^{\mathsf{qf}}_{k+n}(T_{\mathbf{C}})$ such that $T_{\mathbf{C}} \vDash q(\overline{x},\overline{y}) \to \psi(\overline{x},\overline{y})$.
- Let $d \leq k$, and let $\theta(\overline{y}) \in S^{\mathsf{qf}}_n(T_{\mathbf{C}})$ such that $\psi(\overline{x};\overline{y}) \wedge \theta(\overline{y})$ is consistent with $T_{\mathbf{C}}$. Then $K^+_d(\psi/\theta)$ is the set of pairs $(\varphi'(\overline{x};\overline{y}), \varphi/\theta)$, where $\varphi = \varphi(x_{i_0}, ..., x_{i_{d-1}}; \overline{y})$, such that if $\mathcal{M} \vDash \varphi(a_{i_0}, ..., a_{i_{d-1}}, \overline{b})$, then:
  - φ' is a quantifier-free-complete type, and $T_{\mathbf{C}} \vDash \varphi'(\overline{x};\overline{y}) \to \psi(\overline{x};\overline{y}) \wedge \varphi(x_{i_0}, ..., x_{i_{d-1}}; \overline{y})$.
  - φ/θ is a d,n-extension problem.
  - If $\mathcal{M} \vDash \varphi'(\overline{a};\overline{b})$, then $i_0 = \min\left\{t < k : a_t \notin \operatorname{acl}(\overline{b})\right\}$, and for $0 < \ell < d$, $i_\ell = \min\left\{i_{\ell-1} < t < k : a_t \notin \operatorname{acl}\left(\overline{b} \cup \left\{a_{i_0}, ..., a_{i_{\ell-1}}\right\}\right)\right\}$.

  We then define $K_d(\psi/\theta) = \left\{\varphi/\theta : (\varphi', \varphi/\theta) \in K^+_d(\psi/\theta)\right\}$.
- Let $\theta(\overline{y}) \in S^{\mathsf{qf}}_n(T_{\mathbf{C}})$ such that $\psi(\overline{x};\overline{y}) \wedge \theta(\overline{y})$ is consistent with $T_{\mathbf{C}}$. Let $d = \dim_{\mathbf{a}}(\psi(\mathcal{M};\overline{b}))$ where $\overline{b} \in \theta(\mathcal{M})$. If $d > 0$, then we define,

$$\gamma(\psi/\theta) = \sum_{\varphi \in K_d(\psi/\theta)} \gamma(\varphi/\theta)$$

  and if $d = 0$ – so that $\psi(\mathcal{M},\overline{b})$ is finite – then we set $\gamma(\psi/\theta) = \left|\psi(\mathcal{M};\overline{b})\right|$.

- We may then define,

$$\mathbb{A}(\psi) = \left\{ (\theta, d, \gamma(\psi/\theta)) : \begin{array}{l} \bar{b} \in M^n, \, \theta = \mathrm{qftp}(\bar{b}), \\ d = \dim_{\mathbf{a}}(\psi(\mathcal{M}; \bar{b})) \end{array} \right\}.$$

We now have the data $\mathbb{A}$ for a 1-dimensional asymptotic class, and our next (and last) task in proving Theorem 2.1 is to pare down the original class to a cofinal subclass $\mathbf{D}$ that is super-robust and 1-dimensional asymptotic via $\mathbb{A}$. The construction is in stages in which we reduce $\mathbf{C}$ to more and more $\mathbb{A}$-asymptotic subclasses. The definitions need for this process are listed in Definition 2.7 and the intersection properties that guarantee the outcome of the process are given in Lemma 2.8 and Corollary 2.9 that immediately follow Definition 2.7.

Definition 2.7. For every $\mathfrak{a} \in \mathbf{C}$, let $\mathbf{C}_{\mathfrak{a}}$ be the class of $\mathfrak{b} \in \mathbf{C}$ such that $\mathfrak{a}$ embeds into $\mathfrak{b}$.

Further, for $0 < m < \omega$, let F_m be the set of partitioned formulas $\psi(x_0, ..., x_{k-1}; y_0, ..., y_{n-1})$ of quantifier-rank $\leq m$ such that $k + n \leq m$, and let $\mathbf{C}(m)$ be the class of all $\mathfrak{a} \in \mathbf{C}$ such that:

- $Th(\mathfrak{a}) \cap T_{\mathbf{C}} \vDash \psi \leftrightarrow \psi^{\mathsf{qf}}$ if $\psi \in F_m$ is not a sentence;
- $\mathfrak{a} \vDash \psi$ whenever $\psi \in F_m \cap T_{\mathbf{C}}$.

Finally, if $0 < \delta < 1$ and $0 < m < \omega$, then let $\mathbf{C}^{\delta}(m)$ be the class of all $\mathfrak{a} \in \mathbf{C}(m)$ such that for each $\psi(\overline{x}; \overline{y}) \in F_m$ and each $(\theta, d, \gamma(\psi/\theta)) \in \mathbb{A}(\psi)$,

$$\left| \left| \psi(\mathfrak{a}; \bar{b}) \right| - \gamma(\psi/\theta) |\mathfrak{a}|^d \right| \leq \delta |\mathfrak{a}|^d$$

for every $\bar{b} \in \theta(\mathfrak{a})$.

Lemma 2.8. *For all $0 < m < \omega$, $0 < \delta < 1$, and $\mathfrak{a} \in \mathbf{C}$, the class $\mathbf{C}_{\mathfrak{a}} \cap \mathbf{C}^{\delta}(m)$ is non-empty.*

Proof. Let $0 < m < \omega$, $0 < \delta < 1$, and $\mathfrak{a} \in \mathbf{C}$ be given. First of all, we can replace m with $m_1 = \max\{m, |\mathfrak{a}| + 1\}$ so that $\exists x_0 ... x_{|\mathfrak{a}|-1} \theta_{\mathfrak{a}}(\overline{x})$ is in $F_{m_1} \supseteq F_m$, where $\theta_{\mathfrak{a}}$ is a quantifier-free formula asserting the isomorphism type of $\mathfrak{a}$ relative to some enumeration $a_0, ..., a_{|\mathfrak{a}|-1}$ of its universe. Then $\mathbf{C}^{\delta}(m_1) \subseteq \mathbf{C}_{\mathfrak{a}} \cap \mathbf{C}^{\delta}(m)$, and it is enough to show that $\mathbf{C}^{\delta}(m_1)$ is non-empty.

For each non-sentence $\psi(\overline{x}; \overline{y}) \in F_m$, there is an N_{ψ} such that for all $(\theta, d, \gamma(\psi/\theta)) \in \mathbb{A}(\psi)$, if $N_{\psi} \leq N$, then

$$\mu_N \left[\bigvee_{\bar{b} \in N^{|\overline{y}|}} \theta(\bar{b}) \wedge \left| \left| \psi(*; \bar{b}) \right| - \gamma(\psi/\theta) N^d \right| \geq \delta N^d \right] \leq q(N) c^N$$

for an appropriate $0 < c = c_{\psi/\theta} < 1$ and $q = q_{\psi/\theta}(x) \in \mathbb{N}[x]$. We choose N_{m_1} large enough to ensure that:

- $N \geq N_{m_1} \Rightarrow \mu_N \left[\mathbf{C}_N \cap \mathbf{C}(m_1) \right] > 3/4$
- $N_{m_1} \geq N_{\psi}$ for each non-sentence in $\psi(\overline{x}, \overline{y}) \in F_{m_1}$
- $N \geq N_{m_1} \Rightarrow \sum_{\psi \in F_{m_1}} \sum_{(\theta, d, m) \in \mathbb{A}(\psi)} q_{\psi/\theta}(N) c^N_{\psi/\theta} < 1/4$

Then, if $N \geq N_{m_1}$, then $\mu_N \left[\mathbf{C}_N \cap \mathbf{C}^{\delta}(m_1) \right] \geq 1/2 > 0$, so $\mathbf{C}^{\delta}(m_1)$ is non-empty – as required. □

Corollary 2.9. *The family $\{\mathbf{C}_{\mathfrak{a}} : \mathfrak{a} \in \mathbf{C}\} \cup \left\{ \mathbf{C}^{\delta}(m) : m < \omega, \, 0 < \delta < 1 \right\}$ has the finite intersection property.*

Proof of Theorem 2.1. First, we choose a fast-growing, strictly-increasing function $f : \omega \to \omega$, an enumeration $\mathfrak{b}_0, ..., \mathfrak{b}_n, ...$ of the isomorphism types of all members of $\mathbf{C}$, and a decreasing function $\delta : \omega \to (0,1) : k \mapsto \delta_k$ such that $\lim_{k\to\infty} \delta_k = 0$. Then, proceed as follows:

- $\mathbf{D}_0 = \mathbf{C}$;
- If $k > 0$, then $\mathbf{D}_k =$

$$\left\{\mathfrak{a} \in \mathbf{D}_{k-1} : |\mathfrak{a}| \geq f(k) \Rightarrow \mathfrak{a} \in \mathbf{C}_{\mathfrak{b}_k} \cap \mathbf{C}^{\delta_k}(k)\right\}.$$

In the "end," we set $\mathbf{D} = \bigcap_{k<\omega} \mathbf{D}_k$. We take it as obvious that $\mathbf{D}$ is a cofinal sub-class of $\mathbf{C}$. To verify that $\mathbf{D}$ is super-robust, we just observe that for each $\varphi(\overline{x}) \in \mathscr{L}$, if $\mathfrak{a} \in \mathbf{D}$ is sufficiently large, then $\mathfrak{a} \models \forall \overline{x} \left(\varphi(\overline{x}) \leftrightarrow \varphi^{\mathsf{qf}}(\overline{x})\right)$.

Finally, we claim that $\mathbf{D}$ is a 1-dimensional asymptotic class via $\mathbb{A}$ as defined above. Let $\psi(\overline{x};\overline{y}) \in \mathscr{L}^p$, $0 < \varepsilon < 1$, and $(\theta(\overline{y}), d, \gamma(\psi/\theta)) \in \mathbb{A}(\psi)$. We choose $k < \omega$ such that $\delta_k < \varepsilon$. Then for any $\mathfrak{a} \in \mathbf{D}$, if $|\mathfrak{a}| \geq f(k)$, then $\left|\left|\psi(\mathfrak{a};\overline{b})\right| - \gamma(\psi/\theta)|\mathfrak{a}|^d\right| \leq \delta_k|\mathfrak{a}|^d < \varepsilon|\mathfrak{a}|^d$ – and this completes the proof of the theorem. □

The last task of this paper is to prove Theorem 2.2. As already noted it follows quickly as a corollary of Theorems 1.7, 1.18, and 2.1.

Proof of Theorem 2.2. By Theorem 2.1, let $\mathbf{D}$ be a super-robust cofinal sub-class of $\mathbf{C}$ that is 1-dimensional asymptotic. Let $\mathfrak{d}_0, ..., \mathfrak{d}_n, ...$ be pairwise non-isomorphic members of $\mathbf{D}$ that exhaust all isomorphism types in $\mathbf{D}$, and let $\mathscr{U}$ be a non-principal ultrafilter on ω. Since $\mathbf{D}$ is a cofinal super-robust subclass of $\mathbf{C}$, by Theorem 1.7, we have $Th\left(\prod_n \mathfrak{d}_n/\mathscr{U}\right) = T_{\mathbf{D}} = T_{\mathbf{C}}$. Since $\mathbf{D}$ is 1-dimensional asymptotic, by Theorem 1.18, $T_{\mathbf{C}} = Th\left(\prod_n \mathfrak{d}_n/\mathscr{U}\right)$ is super-simple, and $D(X) \leq \dim_{\mathbf{a}}(X)$ for every definable set X of $\prod_n \mathfrak{d}_n/\mathscr{U}$ (which elementarily embeds $\mathcal{M}$) – and this completes the proof of the theorem. □

3. Concluding remarks

To conclude this paper, we make two conjectures which may help to motivate some future work in this area. For the first conjecture, we observe that the 1-dimensionality of the 1-dimensional asymptotic class recovered in Theorem 2.1 seems to come from the role of $\lceil \delta N \rceil$ in the formulation of almost-independent sampling. To account from K-dimensional asymptotic classes with $K > 1$, it seems, that we would need a yet-more relaxed version of independent samplings as follows, perhaps.

Definition 3.1. Let μ be an asymptotic probability measure for $\mathbf{C}$. We say that μ *has broadly almost-independent sampling* if there is a integer $0 < K < \omega$ such that for every $1,n$-extension problem φ/θ of $\mathbf{C}$, there are numbers $0 \leq \varepsilon < 1$, $0 < \delta \leq 1$, and $k \leq K$ such that almost every N, for every $\overline{b} \in N^{(n)}$, for any $\{i_0 < \cdots < i_{t-1}\} \subseteq N \setminus \overline{b}$ where $t = \lceil \delta N^{k/K} \rceil$, for every $s : t \to 2$,

$$\mu_N\left[\bigwedge_{j<t} \varphi(i_j, \overline{b})^{s(j)} \,\middle|\, \theta(\overline{b})\right] \in (1 \pm \varepsilon) \cdot \prod_{j<t} \mu_N\left[\varphi(i_j, \overline{b})^{s(j)} \,\middle|\, \theta(\overline{b})\right].$$

Conjecture 3.2. *Let* $\mathbf{C}$ *be a geometric Fraïssé class with generic model* $\mathcal{M}$*, and suppose that* $\mathbf{C}$ *has the 0,1-law for first-order logic relative to an asymptotic probability measure* μ *that has* *broadly almost-independent sampling. Then for some K, $\mathbf{C}$ contains a cofinal super-robust sub-class $\mathbf{D}$, which is a K-dimensional asymptotic class via $\mathbb{A}$, with*

$$\mathbb{A}(\varphi(\overline{x};\overline{y})) \subset_{fin} \left\{(\mathrm{qftp}(\overline{a}), k/K, m) : \overline{a} \in M^{|\overline{y}|},\ k \leq K{\cdot}\dim_{\mathbf{a}}\big(\varphi(\mathcal{M};\overline{a})\big),\ m \in [0,\infty)\right\}.$$

Consequently, $T_{\mathbf{C}}$ is super-simple, and for any definable set X in $\mathcal{M} \models T$, $D(X) \leq K{\cdot}\dim_{\mathbf{a}}(X)$.

Unfortunately, there is a new asymmetry in broadly almost-independent sampling: while $t = \lfloor \delta N^{k/K} \rfloor$ is adequate, this does not always preclude, say, $t' = \lfloor \delta' N^{(2k+1)/2K} \rfloor$ from also being adequate. Thus, the techniques used in this paper are not alone sufficient to prove something like Conjecture 3.2.

On the other hand, it is not easy (if at all possible) to come up with a geometric Fraïssé class that has the 0,1-law for first-logic, is super-simple, and so forth, without building a 1-dimensional asymptotic class along the way. It would be quite interesting, if not totally unexpected, if the following were found to be true.

Conjecture 3.3. *Suppose $\mathbf{C}$ has the 0,1-law for first-order logic relative to some asymptotic probability measure μ. Further, suppose that $T_{\mathbf{C}}$ is geometric, super-simple, and for every definable set X of the generic model $\mathcal{M}$ of $\mathbf{C}$, $D(X)$ is bounded by the algebraic dimension of X. Then there is an asymptotic probability measure μ' with almost-independent sampling, relative to which $\mathbf{C}$ has the 0,1-law for first-order logic.*

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